OTHER SOLUTION OF FUZZY EQUATION WITH EXTENDED OPERATION

by

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Abstract. In this paper we study other solution of fuzzy equation with extended operation of Sanchez [9].

1. INTRODUCTION

A generalization of classical boolean equation theory consists in fuzzy relational equations theory. IN [7] Sanchez introduced in a brouwerian lattice the concept of max-min fuzzy relational equation in order to give a description of fuzzy systems by equations of fuzzy relations. Zadeh [11] firstly introduced the idea of fuzzy system. After the pioneering papers on fuzzy relational equations theory of Sanchez [7] [8] several authors have studied this theory, for instance: Czogala, Drewniak and Pedrycz [2] , Di Nola [3] [4] , Di Nola and Ventre [5].

Let L be a brouwerian lattice and $F(X)=\{A:X\longrightarrow L\}$ the set of all fuzzy sets of a non empty set X in the sense of Goguen [6]. Let Y,Z be other non empty sets, $Q\in F(X\times Y)$, $R\in F(Y\times Z)$ and $T\in F(X\times Z)$ fuzzy relations.

A max-min fuzzy relational equation is an equation of the following type

$$R \circ Q = T \qquad (1)$$

where "o" is the max-min composition and R is unknown. Let $\mathcal{R}$ be the set of all solution R of equation (1). In [7] Sanchez gives a method of resolution of (1), by determining, if $\mathcal{R}$ is non empty, the greatest element of $\mathcal{R}$, when L is a complete brouwerian lattice. In [3] Di Nola characterizes completely $\mathcal{R}$ when L is a linear lattice. In [9] Sanchez, assuming that * is any operation defined on a product set XxY and taking values on a set Z, and extending it to fuzzy sets by means of Zadeh's extension principle, shows how to solve equation

$$A * B = C \qquad (2)$$

when fuzzy subset A of X and C of Z are given and fuzzy subset B of Y is unknown. In this paper we determine some solution of (2) different from the Sanchez'one. Let us recall some definitions.

2. THE EXTENDED * OPERATION

Let * be an operation defined on a product set XxY and taking values on a set Z. To all (x,y) in XxY, * associates an element z, in Z, which is denoted x*y. The * operation can be extended to fuzzy sets by means of the extension principle (Zadeh, 1975).

Definition 1. Let A be a fuzzy subset of X and B be a fuzzy subset of Y, then the extension principle allows to define a fuzzy subset C = A * B of Z as follows

$$\forall z \in Z, \mu_{A * B}(z) = \sup_{\substack{x \in X, y \in Y \\ x*y = z}} \mu_A(x) \wedge \mu_B(x) \qquad (3)$$

where, as usual, μ_A is the membership function of a fuzzy set A and "$\wedge$" denotes the min operation. Assume first that B is a (non fuzzy) singleton indentified with its unique element, say b in Y, so that $\mu_B(y) = 1$ if y=b and $\mu_B(y) = 0$ if y=b. Thus, equation (3) yields

$$\forall z \in Z, \mu_{A * b}(z) = \sup_{\substack{x \in X \\ x*b=z}} \mu_A(x) \qquad (4)$$

Hence, (3) equivalent to (4')

$$\forall z \in Z,\ \mu_{A * B}(z) = \sup_{y \in Y} \mu_{A * y}(z) \wedge \mu_B(y) \qquad (4')$$

Analogously, exchanging the roles of A and B, we have

$$\forall z \in Z,\ \mu_{A * B}(z) = \sup_{x \in X} \mu_A(x) \wedge \mu_{B * x}(z)$$

3. THE α OPERATOR

In order to solve the * -equation problem on fuzzy sets we need to recall the definition of the α operator which is characteristic of Brouwerian lattices. Given a and b in 0,1 , a α b is defined as the greatest element x in [0,1] , such that $a \wedge x \leq b$, i.e.

$$a \,\alpha\, b = 1 \qquad \text{if } a \leq b$$

$$a \,\alpha\, b = b \qquad \text{if } a > b$$

Here are some properties of the α operator that will be used in the sequel. We recall that, as usual, '$\vee$' denotes the <u>max</u> operation. For all a,b in [0,1] and for all family $(bj)_{j \in I}$ of elements of [0,1] we have

i) $a \wedge (a \alpha b) \leq b$

ii) $a \,\alpha\, (\sup_{i \in I} bi) \geq a \,\alpha\, b_j, \quad \forall b_{j \in I}$

iii) $a \,\alpha\, (a \wedge b) \geq b$

4. * EQUATIONS AND o-EQUATIONS

<u>Notation</u>

From now on, F(U) will denote the class all the fuzzy subsets of a set U.

<u>Definition 2.</u> Given $A \in F(X)$, $C \in F(Z)$ and $*: X x Y \longrightarrow Z$ we define $\overset{\frown}{*}: F(Z) x F(X) \longrightarrow F(Y)$ as follows

$$\forall y \in Y,\ \mu_{C \overset{\frown}{*} A}(y) = \inf_{x \in X, z \in Z} \mu_A(x) \,\alpha\, \mu_C(z)$$

For every pair of fuzzy sets $A \in F(X)$ and $C \in F(Z)$, and for $*: X \times Y \longrightarrow Z$ we have [9]

$$A * (C \check{*} A) \leq C .$$

In other terms, $C \check{*} A$ is a particular solution to $A * X = C$. Moreover $A * X = C$ has a solution if, and only if,

$$A * (C \check{*} A) = C$$

Definition 3. Given $A \in F(X)$ and $*: X \times Y \longrightarrow Z$, we define a fuzzy relation, $(A*)$, from Y to Z, i.e., a fuzzy subset of $Y \times Z$, by its membership function

$$\forall (y,z) \in Y \times Z, \ \mu_{A*}(y,z) = \mu_{A*y}(z) \qquad (5)$$

where $A * y$ is defined in (4).

From (5), $A * B$ can be given by the following expression

$$\forall z \in Z, \ \mu_{A * B}(z) = \sup_{y \in Y} \ \mu_B(y) \wedge \mu_{(A*)}(y,z) = \mu_{(A*) \, o B}(z)$$

Adapting now Sanchez (1976), we derive the following result. Given $C \in F(Z)$, $*: X \times Y \longrightarrow Z$ and $(A*) \in F(Y \times Z)$ we have

$$\mathcal{B} = \{ B \in F(Y) \, / \, (A*) o B = C \} \neq \emptyset \qquad (6)$$

if, and only if, $S = (A*) \, ⓐ \, C \in \mathcal{B}$; then it is the greatest element in ,

where

$$\forall y \in Y, \ \mu_{(A*) ⓐ C}(y) = \inf_{z \in Z} \ \mu_{(A*)}(y,z) \ \alpha \ \mu_C(z)$$

5. OTHER SOLUTION To *-EQUATION

The following theorem is merely an observation, but it plays a key role in successive results.

Theorem I. Let $B', B'' \in \mathcal{B}$ and $B' \in F(Y)$ is such that $B' \leq B \leq B''$. Then $B \in \mathcal{B}$

Theorem II. If $B \notin \mathcal{B}$ and $B' \in F(Y)$ is such that $B' \leq B$ then $B' \notin \mathcal{B}$

Proof If $B' \in \mathcal{B}$, then we should have

$$B' \leq B \leq S$$

that implies $B \in \mathcal{B}$. The contradiction obtained proves the thesis.

Theorem III. If $\mathcal{B} \neq \emptyset$, then we have for all $x \in X$, $z \in Z$, $B \subset \mathcal{B}$

$$\mu_C(z) \leq_{\substack{x\in X \\ x*y=z}} \mu_{A*}(y,z) \text{ and } \mu_C(z) \leq \bigvee_{y\in Y} \mu_B(y)$$

Proof It was already pointed out as a result of (6)

Theorem IV. If there exists some $x \in X$ such that: $\mu_{A*}(y,z) < \mu_C(z)$ for every $y \in Y$, $z \in Z$ and $\mu_C(z') \neq \mu_C(z'')$ for some z', $z'' \in Z$ then $\mathcal{B} = \emptyset$.

Proof If $\mathcal{B} \neq \emptyset$, from theorem III then it is

$$\mu_C(z') \leq \bigvee_{\substack{x\in X \\ x*y=z}} \mu_{A*}(y,z) \leq \mu_C(z')$$

and

$$\mu_C(z'') \leq_{\substack{x\in X \\ x*y=z}} \mu_{A*}(y,z) \leq \mu_C(z'')$$

and this should give

$$\mu_C(z') = \bigvee_{\substack{x\in X \\ x*y=z}} \mu_{A*}(y,z) = \mu_C(z'')$$

in opposition to the hypothesis.

Theorem V. If there exists some x X such that

$$\mu_B(y) < \mu_C(z) \ \forall\ y \in Y,\ z \in Z \quad \text{and} \quad \bigvee_{y\in Y} \mu_B(y) = \mu_B(y')$$

for some $y' \in Y$, then $\mathcal{B} = \emptyset$.

Proof By ipothesis, we draw $\mu_B(y') < \mu_C(z)$. This inequality and the theorem II assure $\mathcal{B} = \emptyset$.

Now for every $y \in Y$ let us denote Q the fuzzy relation defined by the following membership function

$$(\mu_{(A*)\,@\,C}(y)) \wedge (\bigvee_z \mu_C(z))$$

Then holds the

Theorem VI. $\mathcal{B} \neq \emptyset$ if and only if $Q \in \mathcal{B}$

Proof According to (6) it is $\mathcal{B} \neq \emptyset$, if and only if $(A^*)\ \alpha\ C \in \mathcal{B}$. It is then sufficient to prove that Q is solution of (2), whenever it is $(A^*)\ \alpha\ C$.

Indeed it is

$$\bigvee_{y\in Y} (\mu_B(y) \wedge \mu_{(A^*)\,\textcircled{\alpha}\,C}(y) \wedge (\bigvee_{z\in Z} \mu_C(z))) = (\bigvee_{z\in Z} \mu_C(z)) \wedge (\bigvee_{y\in Y} \mu_B(y) \wedge$$

$$\wedge\, \mu_{(A^*)\,\textcircled{\alpha}\,C}(y)) = (\bigvee_{z\in Z} \mu_C(z)) \wedge \mu_C(z) = \mu_C(z)$$

for the well known properties of a brouwerian lattice.

6. A NUMERICAL EXAMPLE

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0,4	0,2	0,8	0,9	0,7	0,5	0,6	0,3	0,1

$Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
0,2	0,8	0,3	0,5	0,4	0,7	0,6	0,9	0,1

$Z = \{z_1, z_2, z_3\}$

z_1,	z_2,	z_3
0,7	0,6	0,5

Let * usual addition on numbers. Then we have

$\mu_{A*y_1}(z_1) = 0{,}5$ $\qquad \mu_B(y_1) \wedge \mu_{A*y_1}(z_1) = 0{,}2 \wedge 0{,}5 = 0{,}2$

$\mu_{A*y_2}(z_1) = 0$ $\qquad \mu_B(y_2) \wedge \mu_{A*y_2}(z_1) = 0{,}8 \wedge 0 = 0$

$\mu_{A*y_3}(z_1) = 0{,}4$ $\qquad \mu_B(y_3) \wedge \mu_{A*y_3}(z_1) = 0{,}3 \wedge 0{,}4 = 0{,}3$

$\mu_{A*y_4}(z_1) = 0{,}2$ $\qquad \mu_B(y_4) \wedge \mu_{A*y_4}(z_1) = 0{,}5 \wedge 0{,}2 = 0{,}2$

$\mu_{A*y_5}(z_1) = 0{,}3$ $\qquad \mu_B(y_5) \wedge \mu_{A*y_5}(z_1) = 0{,}4 \wedge 0{,}3 = 0{,}3$

$\mu_{A*y_6}(z_1) = 0$ $\qquad \mu_B(y_6) \wedge \mu_{A*y_6}(z_1) = 0{,}7 \wedge 0 = 0$

$\mu_{A*y_7}(z_1) = 0{,}1$ $\qquad \mu_B(y_7) \wedge \mu_{A*y_7}(z_1) = 0{,}6 \wedge 0{,}1 = 0{,}1$

$\mu_{A*y_8}(z_1) = 0$ $\qquad \mu_B(y_8) \wedge \mu_{A*y_8}(z_1) = 0{,}9 \wedge 0 = 0$

$\mu_{A*y_9}(z_1) = 0{,}6$ $\qquad \mu_B(y_9) \wedge \mu_{A*y_9}(z_1) = 0{,}1 \wedge 0{,}6 = 0{,}1$

It is then $\mu_{(A*)oB}(z_1) = 0{,}3$.

Analogously we have $\mu_{(A*)oB}(z_2) = 0{,}3$ and $\mu_{(A*)oB}(z_3) = 0{,}5$ and theorems III, IV, and V hold.

Moreover $\mu_{(A*)\,\text{ⓐ}\,C}(y_1) = 0{,}3$, $\mu_{(A*)\,\text{ⓐ}\,C}(y_2) = 1$, $\mu_{(A*)\,\text{ⓐ}\,C}(y_3) = 0{,}4$ and so on.

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