FUZZY COMPLEX NUMBER

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Since the correspondence between the complex number a+bi and binary number pair (a,b) can be readily established, so that the study of the fuzzy set in a complex field may be transformed to that of the fuzzy set in two-dimensional Euclidean space R^2.

1. The Definition of Fuzzy Complex Number and the Relation with Fuzzy Number on R.

Let the membership degree of complex number a+bi to a certain fuzzy set on complex plane be λ, i.e.

$$\mu(a+bi) = \mu(a,b) = \lambda .$$

If λ is viewed as the membership degree of a+bi as a whole, i.e. to define fuzzy complex number like this, then it is but a common fuzzy number on R^2. In fact, complex number a+bi consists of real part and imaginary part. Hence, λ should be acquired according to certain operation, from each membership degree of real part a and imaginary part b. It is what we called the fundamental assumption.

Let the operator be *, i.e.

$$\mu((a,b)) = *(\mu(a), \mu(b)).$$

For instance, we may define

(1) $\mu((a,b)) = \min(\mu(a), \mu(b))$

(2) $\mu((a,b)) = \frac{1}{2}(\mu(a), \mu(b))$.

The following discussion will be developed under the operator assumption of " min ". The definitions of convex fuzzy set and the Descartes' product A×B of two fuzzy sets A and B on R concerned in this paper are introduced from [1]. Before we give the definition of complex fuzzy number, the relation of fuzzy numbers on R as well as on R^2 should be investigated . Now we give the definitions of fuzzy number and rigorous fuzzy number as follows.

Definition 1[1][2] Fuzzy set A over N-dimensional Euclidean space is fuzzy number if and only if

(1) $\exists ! \, x \in R^n$, such that $\mu_A(x) = 1$

(2) A is a convex fuzzy set.

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If condition (3) is supplemented, it is the rigorous fuzzy number.

(3) $\mu_A(x)$ is continuous, and for any $\alpha \in [0,1]$, $\{x \mid \mu_A(x) = \alpha\}$ can not form the region.

For fuzzy set over R, this condition requires that $\mu_A(x)$ is a continuous and sectionally rigorous monotone function.

The following theorem is valid.

Theorem 1. If A and B are fuzzy numbers over R, then product $A \times B$ is also a fuzzy number over R^2; $A \times B$ is a rigorous fuzzy number if A and B are rigorous fuzzy numbers respectively.

While fuzzy number A and B over R are rigorous fuzzy numbers, the graph of $A \times B$'s membership function is equiped with some interesting features.

(1) Assume that the rigorous fuzzy number A is such that $\mu_A(x_0) = 1$, and $\mu_A(x)$ is rigorously monotone in $(-\infty, x_0]$ and $[x_0, +\infty)$; (see Fig.1) The rigorous fuzzy number B is such that $\mu_B(y_0) = 1$ and $\mu_B(y)$ is rigorously monotone in $(-\infty, y_0]$ and $[y_0, +\infty)$.(see Fig.2) Then, we have $\mu_{A\times B}((x_0, y_0)) = 1$. While the curved surface of function $\mu_{A\times B}$ is cutting by plane $\mu_{A\times B}((x,y)) = \lambda$ (denotes $\mu = \lambda$ for short)$(0 < \lambda < 1)$, the intersecting surface is a rectangle. In addition, the sides of the rectangle parallels to X axis and Y axis respectively.(see Fig.3)

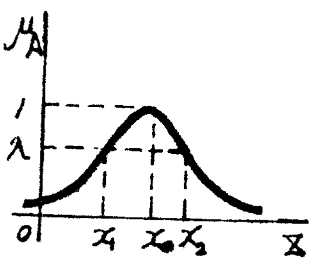

Fig.1

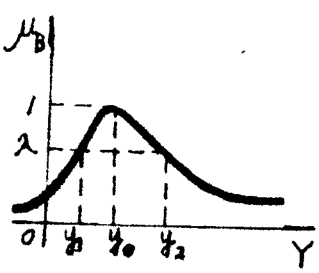

Fig.2

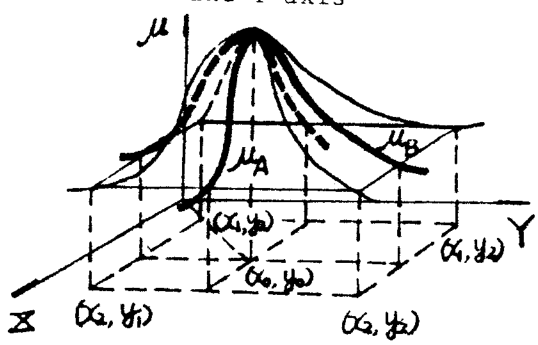

Fig.3

Let $\mu_A(x_1) = \mu_A(x_2) = \lambda$, $\mu_B(y_1) = \mu_B(y_2) = \lambda$, then we have

$$\mu_{A\times B}((x_1,y_1)) = \mu_{A\times B}((x_1,y_2)) = \mu_{A\times B}((x_2,y_1)) = \mu_{A\times B}((x_2,y_2)) = \lambda .$$

In addition, under the perpendicular spatial coordinate system, the four points $(x_1, y_1, \mu_{A\times B}((x_1,y_1)))$, $(x_1, y_2, \mu_{A\times B}((x_1,y_2)))$, $(x_2, y_1, \mu_{A\times B}((x_2,y_1)))$, $(x_2, y_2, \mu_{A\times B}((x_2,y_2)))$ are exactly the four apexes of rectangle.

(2) $\mu_{A\times B}((x,y))$ is cutting by plane $x = x_0$, the intersecting curve is $\mu_B(y)$; whereas cutting by plane $y = y_0$, the intersecting curve is $\mu_A(x)$.

In general, the rigorous fuzzy number Q over R^2 is not necessarily expressed by the Descartes' product of rigorous numbers S, T over R. Even if the restriction of rigorous fuzzy numbers is cancelled, i.e. S and T may be any fuzzy sets, Q is still not necessarily expressed by the form of $S\times T$. However, if the rigorous fuzzy number Q is such that for any $\mu=\lambda\,(0<\lambda<1)$, the intersecting surface of Q cutting by μ is a rectangle, then Q is able to be expressed by $Q = S\times T$, where S, T are rigorous fuzzy number on R. While $\mu_Q((x_0,y_0)) = 1$, S is $\mu_Q((x,y_0))$, T is $\mu_Q((x_0,y))$.

Definition 2. The rigorous fuzzy number over R^2 is defined fuzzy complex number, if for arbitrary $\mu=\lambda(\lambda\in(0,1))$, the intersecting surface of the former cutting by μ is a rectangle.

From the previous analysis we can get

Theorem 2. The Descartes' product of two rigorous fuzzy numbers is fuzzy complex number, whereas any fuzzy complex number can be expressed by the Descartes' product of two rigorous fuzzy numbers.

If it is for the satisfaction of theorem 2 we raise the definition of fuzzy complex number, the interpretion is but a strained one. In fact, under the assumption of $\mu(a+bi) = \min(\mu(a),\mu(b))$, when the figure of rigorous fuzzy number over R^2 is cutting by $\mu=\lambda\,(\lambda\in(0,1))$, the intersecting surface must be rectangle. i.e. The rigorous fuzzy number over R^2 must be fuzzy complex number.

Theorem 3. Under the fundamental assumption of " min ", the rigorous fuzzy number of R^2 certainly be fuzzy complex number.

Proof. Let Q be a rigorous fuzzy number over R^2, (now Q is not necessarily expressed by the Descartes' product of two rigorous fuzzy numbers on R) for any $\lambda\in(0,1)$, what we want to prove is that the intersecting surface of $\mu_Q((x,y))$ cutting by $\mu=\lambda$ is a rectangle.

Let the two points (x_1,y_1) and (x_2,y_2) on R^2 are such that

$\mu_Q((x_1,y_1))\geq\lambda$, $\mu_Q((x_2,y_2))\geq\lambda$, then we have $\mu_Q((x_1,y_2))\geq\lambda$. If not so, from the assumption of "min", $\mu((x_1,y_2))<\lambda$, there should be $\mu(x_1)<\lambda$ or $\mu(y_2)<\lambda$, hence $\mu((x_1,y_1))<\lambda$ or $\mu((x_2,y_2)) < \lambda$. Similarly. we may also get $\mu((x_2,y_1))\geq\lambda$.

Since Q is convex, when (x_1,y_1) and (x_2,y_2) belong to the α-cut Q_α, there exists a rectangle field belonged to Q_α with (x_1,y_1), (x_2,y_2) as the apexes, and composed of line sections which are parallel to two coordinate axes. Hence, if there exists another point (x^*,y^*) such that $\mu_Q((x^*,y^*))\geq\lambda$, we'll get another enlarged rectangle field belonged to Q_α. Eventually, we may find a largest rectangle field, and it is no other than Q_α.

Now let's prove that $\mu_Q(p)=\lambda$, iff point p(x,y) is at the bound of the rectangle field Q_α. If $\exists$ p_1 at the bound of Q_α, and $\mu_Q(p_1)>\lambda$, from the continuity of $\mu_Q((x,y))$, there must be a point p_1^* near p_1 such that $\mu_Q(p_1^*)>\lambda$. Thus, the original rectangle field can be enlarged. It is a contradiction. Therefore, the membership degree of the point at the bound of Q_α must be λ. It is evident that the points outside the rectangle field have a different membership degree from λ . On the other hand, if there exists a point p such that $\mu_Q(p)=\lambda$, then, from the requirement of convexity, there must be a region whose membership degree is λ . It is a contradiction. So the intersecting surface of μ_Q cut by $\mu=\lambda$ must be rectangle. ||

2. Operations of Fuzzy Complex Numbers

The whole fuzzy complex numbers over R is denoted by $\mathcal{FC}(R^2)$. From the principle of extension we now give the following operator $\odot$ in $\mathcal{FC}(R^2)$[1]:

$$\mu_{Q1\odot\ldots\odot Qk}((x,y)) =$$
$$\sup\min(\ _{Q1}((x_1,y_1)),\ \ldots,\ \ _{Qk}((x_k,y_k))).$$
$$(x,y)=(x_1,y_1)*(x_2,y_2)*\ \ldots\ *(x_k,y_k)$$
$$=(x_1*x_2*\ldots*x_k,\ y_1*y_2*\ldots*y_k).$$

Where $Q_i(i=1,2,\ldots k)\in\mathcal{FC}(R^2)$, $*$ is the corresponding operation of $\odot$ in common sence. As an example , we give the law of fuzzy complex number addition.

Let $Q, H \in \mathcal{F}(R^2)$, from the principle of extension, $Q \oplus H$ is defined by

$$\mu_{Q \oplus H}((x,y)) = \sup_{(x,y) = (x_1,y_1) + (x_2,y_2)} \min (\mu_Q((x_1,y_1)), \mu_H((x_2,y_2)))$$

" $\oplus$ " satisfies the following operation laws:

(1) **Commutative law**

$Q \oplus H = H \oplus Q$.

(2) **Associative law**

$(Q \oplus H) \oplus T = Q \oplus (H \oplus T) = Q \oplus H \oplus T.$

Above two laws can be proved directly from the definition of " $\oplus$ ".

It is interesting that, under our definition, the operator " $\oplus$" also possesses the law which is analogous to common complex addition, i.e. add real part to real part and add imaginary part to imaginary part respectively.

(3) Let $Q, R \in \mathcal{F}(R^2)$, and $Q = M \times N$, $R = S \times T$, then

$Q \oplus R = (M \times N) \oplus (S \times T) = (M \circledcirc S) \times (N \circledcirc T)$,

where M, N, S, T are rigorous fuzzy numbers over R; $\circledcirc$ is an operation on R defined by principle of extension.

$$\mu_{M \circledcirc N}(z) = \sup_{z=x+y} \min (\mu_M(x), \mu_S(y))^{[1]}.$$

Proof. Let any $(x,y) \in R^2$, and $x_1^* + x_2^* = x$, x_1^* and x_2^* are located at the monotone intervals of M and S simutaneously, and $\mu_M(x_1^*) = \mu_S(x_2^*) = a$; Similarly, we have y_1^*, y_2^* , and y_1^* y_2^* are located at the monotone intervals of N and T with that $\mu_N(y_1^*) = \mu_T(y_2^*) = b$.

Then

$$\mu_{(M \circledcirc S)\times(N \circledcirc T)}(x,y) = \min (\mu_{M \circledcirc S}(x), \mu_{N \circledcirc T}(y))$$
$$= \min(\sup_{x=x_1+x_2} \min (\mu_M(x_1), \mu_S(x_2)), \sup_{y=y_1+y_2} \min (\mu_N(y_1), \mu_T(y_2)))$$
$$= \min (a,b) \qquad \text{(see[1])}$$

$$\mu_{Q \oplus R}(x,y) = \sup_{(x,y)=(x_1,y_1) + (x_2,y_2)} \min (\mu_Q((x_1,y_1)), \mu_R((x_2,y_2)))$$

$$= \sup_{(x,y) = (x_1,y_1) + (x_2,y_2)} \min(\min(\mu_M(x_1), \mu_S(x_2)), \min(\mu_N(y_1), \mu_T(y_2)))$$

Since, while $x = x_1+x_2$,

$$\min(\mu_M(x_1), \mu_S(x_2)) \leq \min(\mu_M(x_1^*), \mu_S(x_2^*)) = a,$$

$$\min(\mu_N(y_1), \mu_T(y_2)) \leq \min(\mu_N(y_1^*), \mu_T(y_2^*)) = b,$$

hence,

$$\min(\min(\mu_M(x_1), \mu_S(x_2)), \min(\mu_N(y_1), \mu_T(y_2))) \leq \min(a,b).$$

So,

$$\mu_{Q \oplus R}(x,y) = \min(a,b).$$

Therefore,

$$Q \oplus R = (M \times N) \oplus (S \times T) = (M \circledcirc S)\ (N \circledcirc T). \quad ||$$

Theorem 4. $Q \oplus h$ is a fuzzy complex number, if Q and H are fuzzy complex numbers respectively. i.e. $\mathcal{FC}(R^2)$ is closed for fuzzy addition.

Proof.(1) if $\exists!$ (x_1,y_1) such that $\mu_Q((x_1,y_1)) = 1$,

$\exists!$ (x_2,y_2) such that $\mu_H((x_2,y_2)) = 1$,

Obviously, $\exists!$ (x_1+x_2, y_1+y_2) such that $\mu_{Q \oplus H}((x_1+x_2,y_1+y_2)) = 1$.

(2) To prove that $Q \oplus H$ is convex fuzzy set.

Let $Q = M \times N$, $H = S \times T$, M, N, S, T be rigorous fuzzy numbers over R. From law (3) we have $Q \oplus H = (M \circledcirc S)\times(N \circledcirc T)$, whereas $M \circledcirc S$ and $N \circledcirc T$ are convex sets (see [1]), according to theorem 1 we know that the Descartes' product of them is also a convex fuzzy set.

(3) To prove the continuity.

Follow the proof of convexity, in accord with the continuity of generalized sum of continuous fuzzy numbers and theorem 1, we can easily know that the continuity is valid.

(4) From the compatibility principle of extension principle with regard to α-cut set[1];

$$(Q \oplus H)_\alpha = Q_\alpha \oplus H_\alpha,$$

since Q_α and H_α are rectangle field, then $(Q \oplus H)_\alpha$ is also a rectangle field.(from Fig.4 we can see that $A_1 + A_2 = A$, $B_1 + B_2 = B$, $C_1 + C_2 = C$, $D_1 + D_2 = D$. and the length of side of the new rectangle field equals to the sum of two relative sides of original rectangle fields.)

For any p(x,y) in the rectangle field, $\exists\ p_1(x_1,y_1) \in Q_\alpha$, $p_2(x_2,y_2) \in H_\alpha$, and $p(x,y) = p_1(x_1,y_1) + p_2(x_2,y_2)$. Thus $\mu_{Q \oplus H}(p(x,y)) > \alpha$. It is because that the point which is located at the bound of the rectangle field should have the membership degree α, therefore the intersecting surface whose hight is α should be a rectangle and $\{(x,y) \mid \mu_{Q \oplus H}((x,y)) = \alpha\}$ can not form a region.

As a conclusion of (1),(2),(3),(4), $Q \oplus H$ must be fuzzy complex number. ||

Of cause, we may define other operators for fuzzy complex numbers, and yet the special operators may possess particular regulations which are waiting for us to make a further study.

REFERENCES

[1] D. Dubois and H. Prade, Fuzzy Sets and Systems : Theory and Application (Academic Press, New York, 1980)

[2] Wan Peizhuang, Fuzzy Sets Theory and Applications, Publishing Hause of Shanghai Science & Technology, 1983.

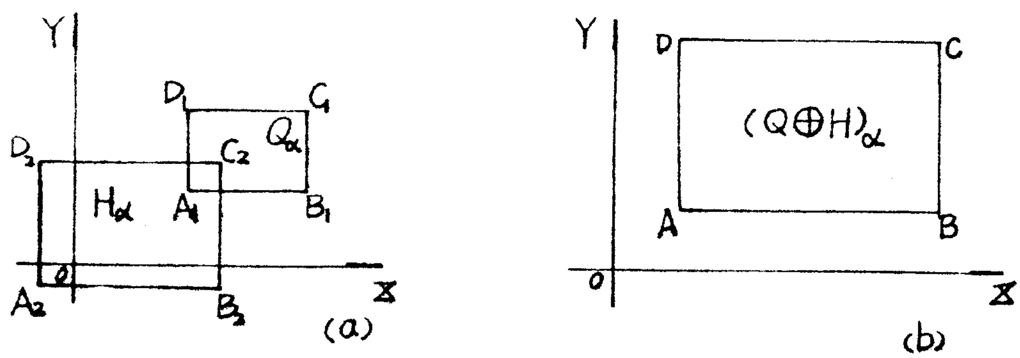

Fig. 4

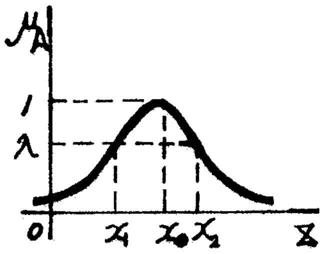

Fig. 1

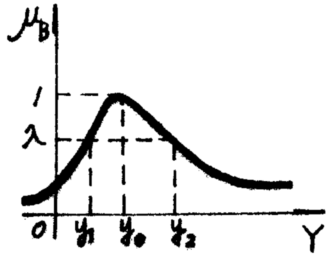

Fig. 2

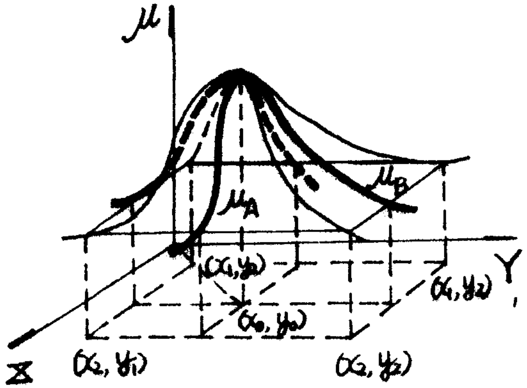

Fig. 3

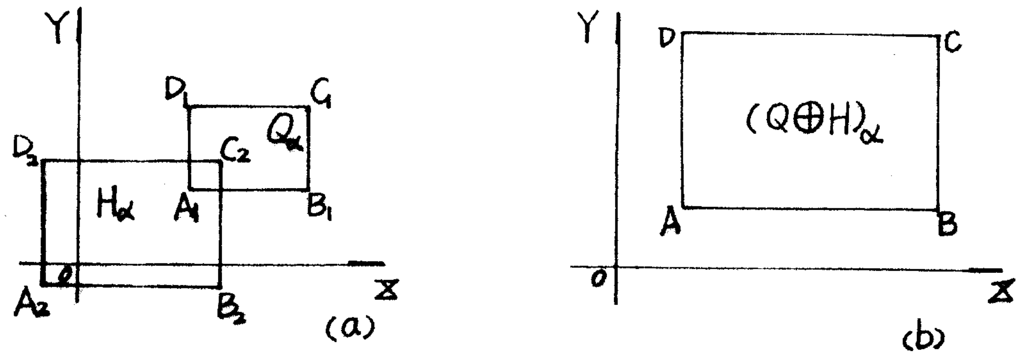

Fig. 4