# T-NORMS AND THEIR PICTORIAL REPRESENTATIONS

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This paper shows a number of examples of existing t-norms which are known as a good model of a fuzzy set-theoretic intersection, and then the pictorial representations of t-norms are made with the aid of computer. Moreover, several new examples of t-norms are proposed.

The discussion of t-conorms (the dual operations of t-norms) is omitted because of limitation of space.

## 1. T-NORMS

Triangular norms (t-norms for short) were introduced by Menger [1], and studied extensively by Schweizer and Sklar [2] in the context of statistical metric spaces. There has been recently a consensus to admit the concept of t-norms to represent pointwise fuzzy set-theoretic intersection [3-10].

A function T: $[0,1] \times [0,1] \to [0,1]$ will be called <u>t-norm</u> iff for any $x,y,z \in [0,1]$

(i) $T(x, 1) = x$ (existing of a unit 1)

(ii) $x_1 \leqq x_2 \implies T(x_1, y) \leqq T(x_2, y)$ (monotonicity)

(iii) $T(x, y) = T(y, x)$ (commutativity)

(iv) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)

From an algebraic point of view, a t-norm defines a semigroup on [0,1] with a unit 1 and a zero 0 and the semigroup operation is order preserving and commutative.

A t-norm T will be called <u>Archmedian</u> iff

(v) T is continuous

(vi) $T(x,x) < x$ for all $x \in (0,1)$

An Archmedian t-norm will be called <u>strict</u> iff

(vii) T is strictly increasing in $(0,1)\times(0,1)$

Given a t-norm T one can consider another two-place function $S : [0, 1] \times [0 ,1] \to [0 ,1]$ defined by

$$S(x, y) = 1 - T(1-x, 1-y).$$

S is called a <u>t-conorm</u> (or the dual of T).

Every Archimedian t-norm is representable by a continuous and decreasing function f from [0,1] into [0,∞] with f(1) = 0 such that

$$T(x,y) = f^{[-1]}(f(x) + f(y))$$

where $f^{[-1]}$ is the pseudeinverse of f, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \cdots\ y \in [0, f(0)] \\ 0 & \cdots\ y \in [f(0), \infty] \end{cases}$$

The function f is called <u>additive generator</u> of T. It is unique except for multiplication with positive numbers. In the non-strict case (that is, <u>nilpotent</u> Archimedian t-norm) we will call the additive generator with f(0) = 1 the <u>normed generator</u>.

There are many examples of t-norms of which we list the most interesting ones with its additive generators in Table 1. Note that ① and ⑥ are not Archimedian t-norms and thus there exist no additive generators for them. The functions f written in ① and ⑥ are considered as additive functions in limited cases. ② - ④ are strict t-norms and ⑤ is a nilpotet t-norm.

Table 1 List of T-Norms and Their Additive Generators

| | | |
|---|---|---|
| ① | $x \wedge y = \min \{x, y\}$ | 【logical product】 |
| | $f(x) = \begin{cases} 1 & \cdots\ x = 0 \\ 0 & \cdots\ x > 0 \end{cases}$ (hard negation) | |
| ② | $x \boxdot y = \dfrac{xy}{x + y - xy}$ | 【Hamacher product】 |
| | $f(x) = \dfrac{1 - x}{x}$ | |
| ③ | $x \cdot y = xy$ | 【algebraic product】 |
| | $f(x) = -\log x$ | |
| ④ | $x \mid\cdot\mid y = \dfrac{xy}{1 + (1-x)(1-y)}$ | 【Einstein product】 |
| | $f(x) = \log \dfrac{2 - x}{x}$ | |
| ⑤ | $x \odot y = 0 \vee (x + y - 1)$ | 【bounded product】 |
| | $f(x) = 1 - x$ (negation) | |
| ⑥ | $x \overset{\wedge}{\cdot} y = \begin{cases} x & \cdots\ y = 1 \\ y & \cdots\ x = 1 \\ 0 & \cdots\ x,y < 1 \end{cases}$ | 【drastic product】 |
| | $f(x) = \begin{cases} 1 & \cdots\ x < 1 \\ 0 & \cdots\ x = 1 \end{cases}$ (soft negation) | |

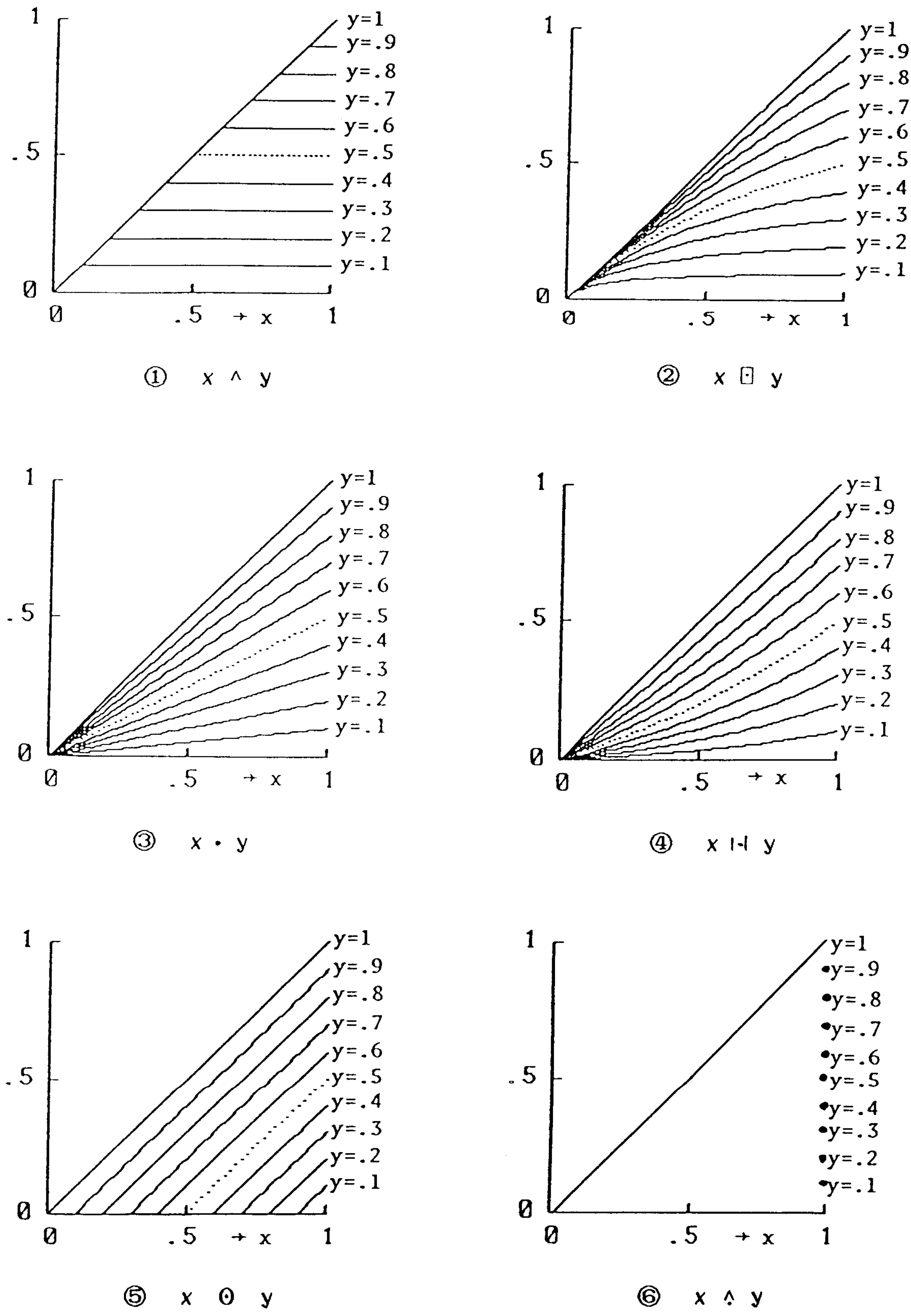



Fig. 1 T-Norms in Table 1

The pictorial representations of these t-norms as parameter y are made in Fig. 1. From the figures we can indicate the ordering relation of these t-norms as shown in Fig.2. It is found that $\wedge$ is the greatest t-norm, while $\dot{\wedge}$ is the least t-norm. The algebraic properties of these t-norms are found in [5, 8].

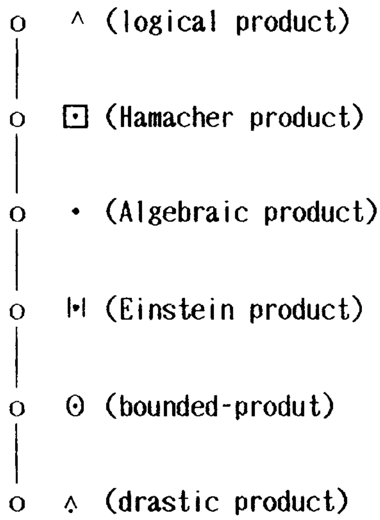



Fig.2 Ordering of t-norms in Table 1.

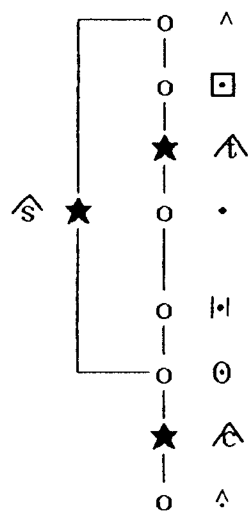
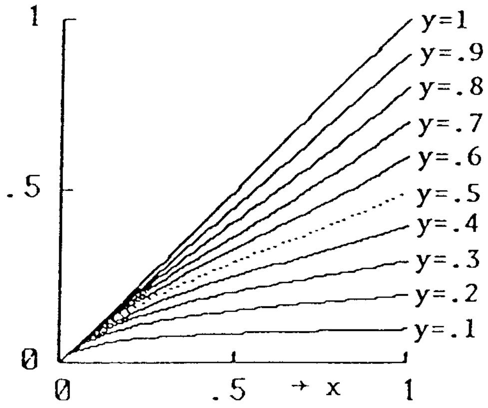


Fig. 3 Ordering of new t-norms in Table 2.

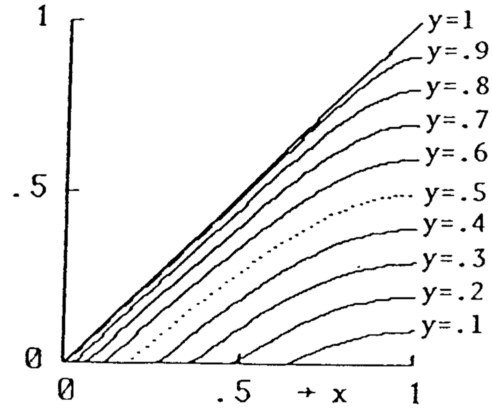
We shall next introduce new t-norms in Table 2. All of them are t-norms generated by additive generators using triangular functions. ⑦ is a strict t-norm, and ⑧ and ⑨ are nilpotent t-norms. The ordering of these t-norms are in Fig. 3 and their pictorial representations are in Fig.4.

Table 2 New T-Norms

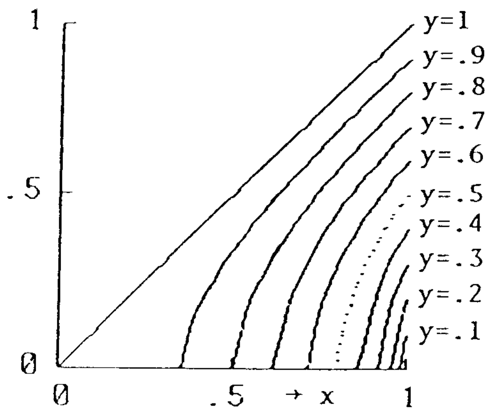
| | |
|---|---|
| ⑦ | $x \overset{\wedge}{t} y = \frac{2}{\pi}\cot^{-1}[\cot\frac{\pi}{2}x + \cot\frac{\pi}{2}y]$<br>$f(x) = \cot\frac{\pi}{2}x$<br>$x \cdot y \leqq x \overset{\wedge}{t} y \leqq x \boxdot y$ |
| ⑧ | $x \overset{\wedge}{s} y = \frac{2}{\pi}\sin^{-1}[(\sin\frac{\pi}{2}x + \sin\frac{\pi}{2}y - 1) \vee 0]$<br>$f(x) = 1 - \sin\frac{\pi}{2}x$<br>$x \odot y \leqq x \overset{\wedge}{s} y \leqq x \wedge y \quad (\overset{\wedge}{s} \neq \vert\cdot\vert, \cdot, \boxdot)$ |
| ⑨ | $x \overset{\wedge}{c} y = \frac{2}{\pi}\cos^{-1}[(\cos\frac{\pi}{2}x + \cos\frac{\pi}{2}y) \wedge 1]$<br>$f(x) = \cos\frac{\pi}{2}x$<br>$x \dot{\wedge} y \leqq x \overset{\wedge}{c} y \leqq x \odot y$ |




⑦ $x \,\widehat{t}\, y$




⑧ $x \,\widehat{s}\, y$




⑨ $x \,\widehat{c}\, y$

Fig. 4 New T-Norms in Table 2

## 2. PARAMETERIZED T-NORMS

In the following, we shall show t-norms with parameter p in Table 3. The t-norms of ⑫ - ⑯ are all strict t-norms, while ⑩, ⑪ and ⑰ are nilpotent t-norms. The t-norm ⑭ by Hamacher is known as the only strict t-norm which can be expressed as rational functions. Frank's t-norm ⑮ and its co-norm are the only pair which satisfies the property:

$$T(x, y) + S(x, y) = x + y$$

They are also useful in the study of fuzzy $\sigma$-algebras.

Fig. 6 shows the parameterized t-norms at y = 0.7, and in Table 4 we have the relationship of the parameterized t-norms and their parameter p which shows at what value of parameter p the parameterized t-norms coincide with the t-norms in Table 1. It is found from Table 4 that the t-norms of ⑩, ⑬ and ⑯ by Yager, Schweizer(3) and Dombi are the broadest t-norms ranging from the least t-norm $\underset{\cdot}{\wedge}$ to the greatest t-norm $\wedge$ .

Table 3　T-Norms with Parameter p

| | | | |
|---|---|---|---|
| ⑩ | $1 - (1 \wedge \sqrt[p]{(1-x)^p + (1-y)^p}\,) \quad \cdots \quad p > 0$<br>$f(x) = (1-x)^p$ | | 【Yager】 |
| ⑪ | $\sqrt[p]{0 \vee (x^p + y^p - 1)} \quad \cdots \quad p > 0$<br>$f(x) = 1 - x^p$ | | 【Schweizer(1)】 |
| ⑫ | $\dfrac{1}{\sqrt[p]{\dfrac{1}{x^p} + \dfrac{1}{y^p} - 1}} \quad \cdots \quad p > 0$<br>$f(x) = \dfrac{1}{x^p} - 1$ | | 【Schweizer(2)】 |
| ⑬ | $1 - \sqrt[p]{(1-x)^p + (1-y)^p - (1-x)^p(1-y)^p} \quad \cdots \quad p > 0$<br>$f(x) = -\log(1 - (1-x)^p)$ | | 【Schweizer(3)】 |
| ⑭ | $\dfrac{xy}{p + (1-p)(x + y - xy)} \quad \cdots \quad p \geqq 0$<br>$f(x) = \log \dfrac{p + (1-p)x}{x}$ | | 【Hamacher】 |
| ⑮ | $\log_p [\, 1 + \dfrac{(p^x - 1)(p^y - 1)}{p - 1} \,] \quad \cdots \quad p > 0 \quad (p \neq 1)$<br>$f(x) = \log_p \dfrac{p - 1}{p^x - 1}$ | | 【Frank】 |
| ⑯ | $\dfrac{1}{1 + \sqrt[p]{(\dfrac{1-x}{x})^p + (\dfrac{1-y}{y})^p}} \quad \cdots \quad p > 0$<br>$f(x) = (\dfrac{1-x}{x})^p$ | | 【Dombi】 |
| ⑰ | $[\,(1+p)(x + y - 1) - pxy\,] \vee 0 \quad \cdots \quad p > -1$<br>$f(x) = \dfrac{\log(1 + p(1-x))}{\log(1+p)}$ | | 【Weber】 |
| ⑱ | $\dfrac{xy}{x \vee y \vee p} \quad \cdots \quad 0 \leqq p \leqq 1$<br>f(x) = undefined | | 【Dubois】 |

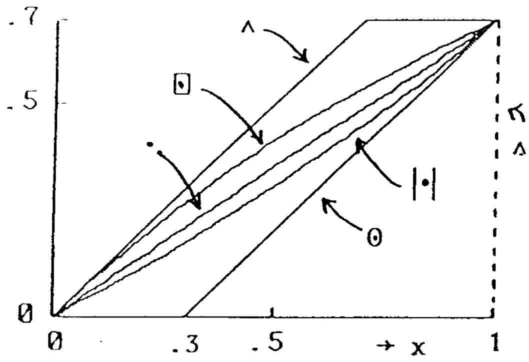



Fig. 5 T-Norms at y = 0.7 in Table 1

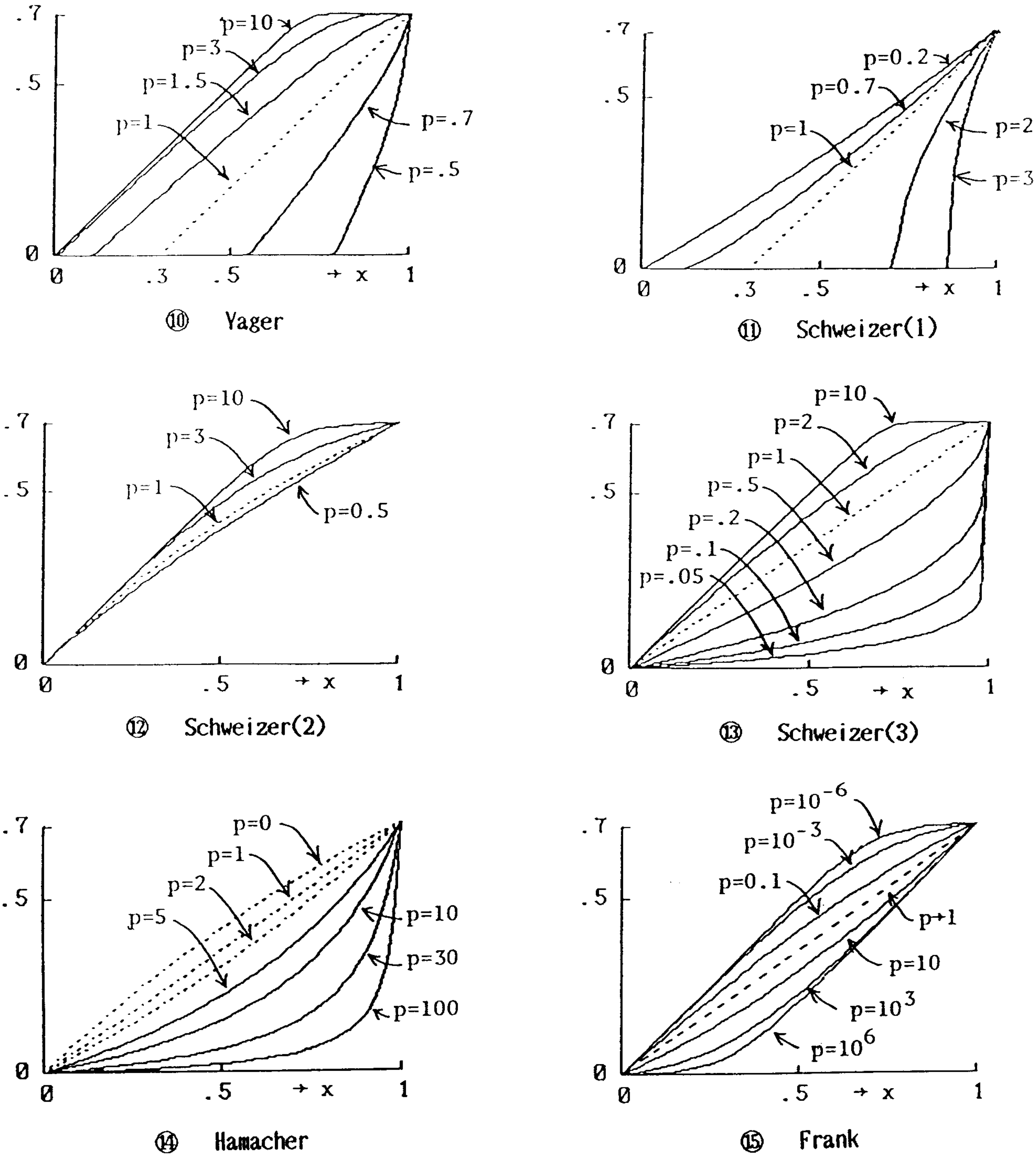
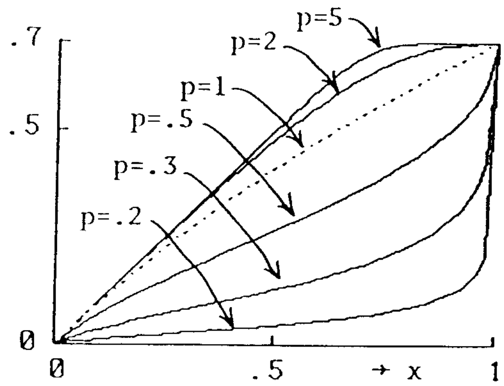
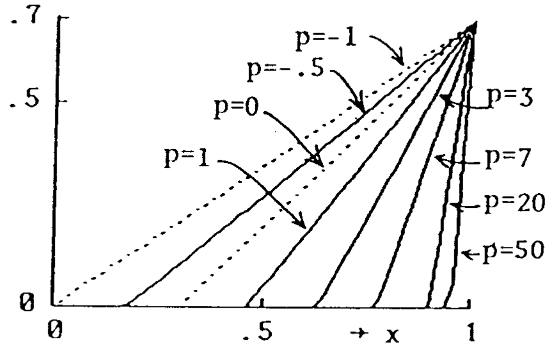


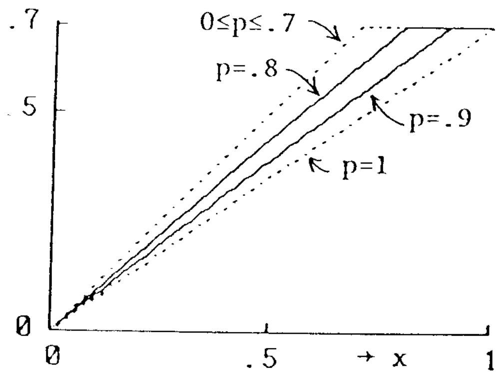

Fig. 6 Parameterized T-Norms at y = 0.7 in Table 3




⑯ Dombi




⑰ Weber




⑱ Dubois

Fig. 6 contined

Table 4 Relationship of Parameterized T-Norms and Their Parameter p

| | $\underset{\cdot}{\wedge}$ | $\odot$ | $\lvert\cdot\rvert$ | $\cdot$ | $\boxdot$ | $\wedge$ |
|---|---|---|---|---|---|---|
| Yager | 0 ← | 1 | | | | ∞ |
| Schweizer(1) | ∞ | 1 | | → 0 | | |
| Schweizer(2) | | | | 0 ← | 1 | ∞ |
| Schweizer(3) | 0 ← | | | 1 | | ∞ |
| Hamacher | ∞ | | 2 | 1 | 0 | |
| Frank | | ∞ | | → 1 | | → 0 |
| Dombi | 0 ← | | | | 1 | ∞ |
| Weber | ∞ | 0 | | - 1 | | |
| Dubois | | | | 1 | | 0 |

## 3. NEW PARAMETERIZED T-NORMS

We shall propose several new parameterized t-norms in Table 5. The t-norms of ⑲ - ㉒ are nilpotent t-norms, and ㉓ - ㉕ are strict t-norms. The t-norm of ⑲ is generated by an additive generator using Sugeno's negation.

Fig. 5 shows the pictorial representation of the t-norms when y = 0.7, and Table 5 indicates the relationship of the t-norms and parameter p.

As the general form of t-norms of ⑳, ㉑ and ㉒, we can show a nilpotent t-norm with three parameters m, n and p, together with its additive generator f(x) and the range of the parameterized t-norm.

$$\sqrt[n]{1 - \frac{1}{m}\log_p \left[ \left(p^{m(1-x^n)} + p^{m(1-y^n)} - 1\right) \wedge p^m \right]} \quad \cdots \quad m, n > 0, \quad p > 1$$

$$f(x) = \frac{1}{p^m - 1}\left( p^{m(1-x^n)} - 1 \right)$$

$$x \underset{\cdot}{\wedge} y \leqq x \mathbin{\hat{\ast}} y \leqq x \wedge y$$

Moreover, we can obtain a strict t-norm with three parameters m, n and p as a general form of ㉓, ㉔ and ㉕ .

$$\frac{1}{\sqrt[n]{\frac{1}{m}\log_p \left( p^{m/x^n} + p^{m/y^n} - p^m \right)}} \quad \cdots \quad m, n > 0, \quad p > 1$$

$$f(x) = p^{m/x^n} - p^m$$

$$x \cdot y \leqq x \mathbin{\hat{\ast}} y \leqq x \wedge y$$

## 4. CONSTRUCTION OF ADDITIVE GENERATORS

Finally, we shall briefly indicate the way of constructing new additive generators of t-norms from a given additive generator. We can show two methods.

[1] Let f(x) be a given normed additive generator, then we can construct a new additive generator f'(x) as follows.

$$f'(x) = 1 - f(1 - x)$$

which is also a normed generator.

For example, when $f(x) = (1 - x)^p$, f'(x) is given as $f'(x) = 1 - x^p$ . These additive generators are ones by Yager ⑩ and Schweizer(1) ⑪.

Table 5 New T-Norms with Parameter p

| | | | |
|---|---|---|---|
| ⑲ | $\frac{1}{p}\left[\frac{1}{\left(\frac{1}{1+px}+\frac{1}{1+py}-\frac{1}{1+p}\right)\vee 1}-1\right] \cdots -1<p<0$ <br> $\frac{1}{p}\left[\frac{1}{\left(\frac{1}{1+px}+\frac{1}{1+py}-\frac{1}{1+p}\right)\wedge 1}-1\right] \cdots p>0$ | $f(x)=\frac{1-x}{1+px}$ (Sugeno's negation) | $\underset{(p\to -1)}{x \dot{\wedge} y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to\infty)}{x \boxdot y}$ |
| ⑳ | $1-\log_p[(p^{1-x}+p^{1-y}-1)\wedge p] \cdots p>1$ | $f(x)=\frac{1}{p-1}(p^{1-x}-1)$ | $\underset{(p\to 1)}{x \odot y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to\infty)}{x \wedge y}$ |
| ㉑ | $1-\frac{1}{p}\log[(e^{p(1-x)}+e^{p(1-y)}-1)\wedge e^{p}] \cdots P>0$ | $f(x)=\frac{e^{p(1-x)}-1}{e^{p}-1}$ | $\underset{(p\to 0)}{x \odot y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to\infty)}{x \wedge y}$ |
| ㉒ | $\sqrt[p]{1-\log[(e^{1-x^{p}}+e^{1-y^{p}}-1)\wedge e]} \cdots p>0$ | $f(x)=\frac{e^{1-x^{p}}-1}{e-1}$ | $\underset{(p\to\infty)}{x \dot{\wedge} y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to 0)}{x \cdot y}$ |
| ㉓ | $\frac{1}{\log_p[p^{1/x}+p^{1/y}-p]} \cdots p>1$ | $f(x)=p^{1/x}-p$ | $\underset{(p\to 1)}{x \boxdot y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to\infty)}{x \wedge y}$ |
| ㉔ | $\frac{1}{\frac{1}{p}\log[e^{p/x}+e^{p/y}-e^{p}]} \cdots p>0$ | $f(x)=e^{p/x}-e^{p}$ | $\underset{(p\to 0)}{x \boxdot y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to\infty)}{x \wedge y}$ |
| ㉕ | $\frac{1}{\sqrt[p]{\log[e^{1/x^{p}}+e^{1/y^{p}}-e]}} \cdots p>0$ | $f(x)=e^{1/x^{p}}-e$ | $\underset{(p\to 0)}{x \cdot y} \leqq x \overset{*}{\wedge} y \leqq \underset{(p\to\infty)}{x \wedge y}$ |

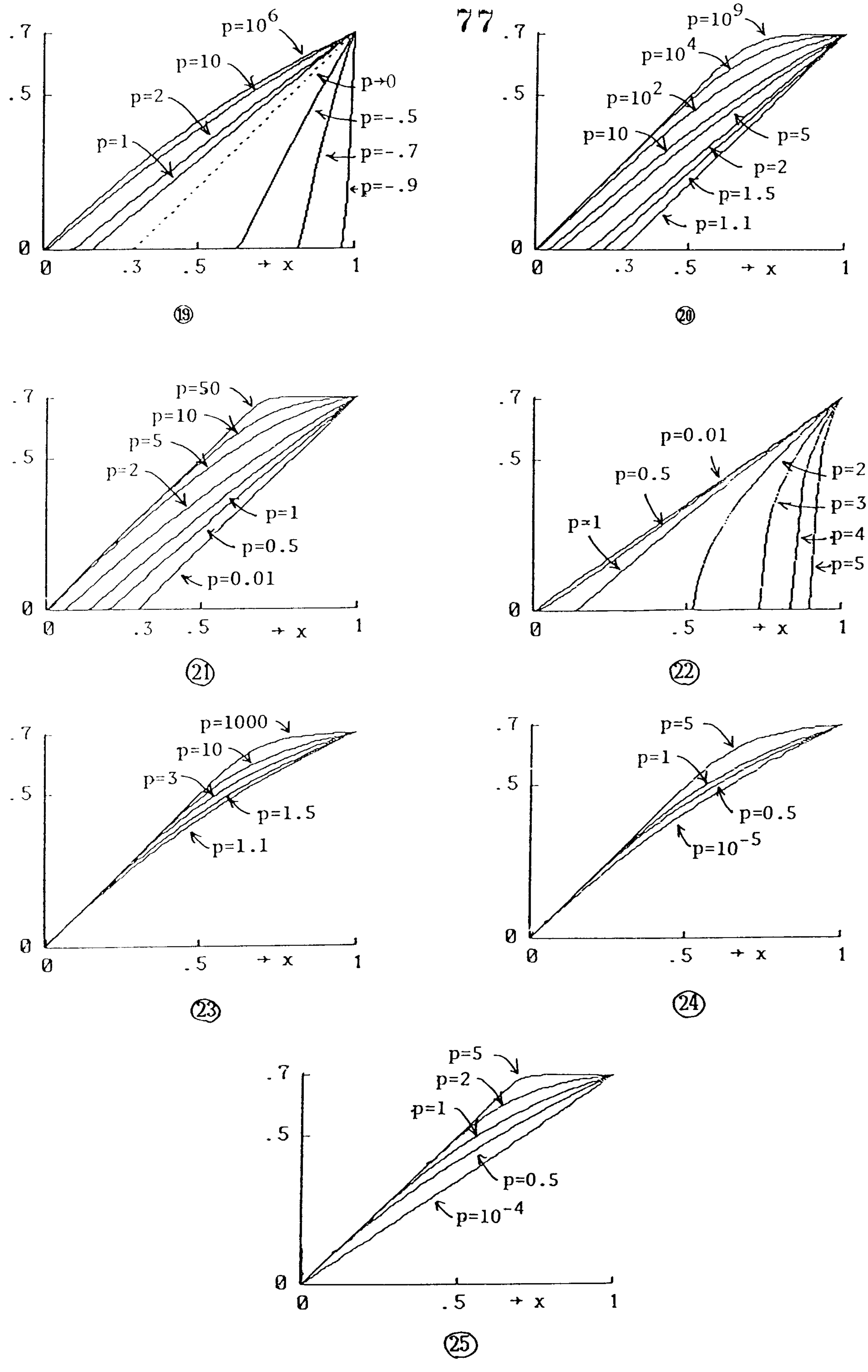



Fig. 7 New Parameterized T-Norms at y = 0.7 in Table 5

Table 6 Relationship of New T-norms in Table 5 and Parameter p

| | $\dot{\wedge}$ | $\odot$ | $\vert\cdot\vert$ | $\cdot$ | $\boxdot$ | $\wedge$ |
|---|---|---|---|---|---|---|
| t-norm ⑲ | -1 ← | 0 ← | | | ∞ | |
| t-norm ⑳ | | 1 ← | | | | ∞ |
| t-norm ㉑ | | 0 ← | | | | ∞ |
| t-norm ㉒ | ∞ | | | → 0 | | |
| t-norm ㉓ | | | | | 1 ← | ∞ |
| t-norm ㉔ | | | | | 0 ← | ∞ |
| t-norm ㉕ | | | | 0 ← | | ∞ |

[2] For any additive generator f(x), we can have new generator by the following.

$$f'(x) = f(g(x))$$

where g(x) is a normed generator which generates a "t-conorm", that is, g(x) is a continuous and increasing function g: $[0,1] \rightarrow [0,1]$ with g(0) = 0 and g(1) = 1.

For example, let f(x) = 1 - x and $g(x) = x^p$, then we have $f'(x) = 1 - x^p$, which are additive generators of bounded-product ⑤ and t-norm by Schweizer(1) ⑪. Moreover, let $g(x) = \sin(\frac{\pi}{2}x)$, then $f'(x) = 1 - \sin(\frac{\pi}{2}x)$ is obtained and it generates new t-norm of ⑧.

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