A COMMON METHOD TO FUZZIFY NONFUZZY CONCEPTS

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1. INTRODUCTION

At present, the fuzzifing of each nonfuzzy mathematical concept is separately completed and one by one defined. These methods are both mechanical and lack reasonable explanation Whether there is a common method to fuzzify nonfuzzy concept is an interesting problem.

This paper gives theorems which make it possible, that the problem is to solve. The method proved in this paper is effective and it shows the general character that the fuzzification is a common concept.

2. DEFINITIONS AND SYMBOLS

A is used to express a fuzzy set on some universes. A(u) is the grade of membership of u in A. λA and $\lambda \cup A$ are fuzzy sets, their grade of membership are $\lambda \wedge A(u)$ and $\lambda \vee A(u)$, respectively. I is the real interval $[0,1]$.

Definition 1. $f(\alpha)$ is called a special set, if the $f(\alpha)$ is a monotone decreasing fuzzy set in I.

3. THEOREMS

Theorem 1. If $f(\lambda)$ is a special set, then

$$\bigvee_{\lambda\in I} \lambda \wedge f(\lambda) = \bigwedge_{\lambda\in I} \lambda \vee f(\lambda).$$

Proof. Clearly, if $f(\lambda) \equiv 0$ or $f(\lambda) \equiv 1$ the theorem is true. If $f(\lambda) \not\equiv 0$, $f(\lambda) \not\equiv 1$, then $f(0)>0$ and $f(1)<1$. Assume $g(\lambda)=f(\lambda)-\lambda$, $g(\lambda)$ is a strict monotone decreasing in I.

$$g(0) = f(0)-0>0,$$
$$g(1) = f(1) -1<0.$$

Hence there exists a single point λ_0 in $[0,1]$, it makes the sign of $g(\lambda)$ in $[0,\lambda_0)$ and $(\lambda_0, 1]$ is opposite. First we consider the situation of $g(\lambda_0)\geq 0$. Since $\lambda\wedge f(\lambda)$ and $\lambda\vee f(\lambda)$ can be written.

$$\lambda\wedge f(\lambda)=\begin{cases} \lambda , & [0, \lambda_0] , \\ f(\lambda), & (\lambda_0, 1] ; \end{cases}$$

$$\lambda\vee f(\lambda)=\begin{cases} f(\lambda), & [0, \lambda_0), \\ \lambda , & (\lambda_0, 1] . \end{cases}$$

Such that the proof of theorem is plain and clear. The same holds for $g(\lambda_0) < 0$.

We assume $H(\lambda)$ is an ordinary subset of universes U , $(\lambda \in I)$. And by [1], if

$$A_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq A_\lambda ,$$

we have

$$A = \bigcup_{\lambda\in I} \lambda A_\lambda \qquad (1)$$

$$A = \bigcup_{\lambda\in I} \lambda A_{\underset{\cdot}{\lambda}} \qquad (2)$$

$$A = \bigcup_{\lambda\in I} \lambda H(\lambda) \qquad (3)$$

In the following we shall give new representation form of Decomposition theorem.

For $\forall u \in U$, $A_\lambda(u)$ is a special set, from the theorem 1 we get

$$A(u) = \bigvee_{\lambda\in I} \lambda \wedge A_\lambda(u) = \bigwedge_{\lambda\in I} \lambda \vee A_\lambda(u) = (\bigcap_{\lambda\in I} \lambda \cup A_\lambda)(u) .$$

Hence, the new Decomposition theorem is obtained immediately

$$A = \bigcap_{\lambda\in I} \lambda \cup A_\lambda \qquad (1')$$

The same we have

$$\underset{\cdot}{A} = \bigcap_{\lambda\in I} \lambda \cup A_{\underset{\cdot}{\lambda}} \qquad (2')$$

$$A = \bigcap_{\lambda\in I} \lambda \cup H(\lambda) \qquad (3')$$

These (1'), (2'), (3') are parallel to (1), (2), (3), respectively. In application, they not only have special effect, but also are complement each other.

Theorem 2. Let $f^\alpha(\lambda)$ be a special set, ($\alpha \in T$, T is index of a set). $\underset{\alpha\in T}{*} f^\alpha(\lambda)$ is the expression of $f^\alpha(\lambda)$, $\vee$, $\wedge$ in accordance with a definite rule. Then

$$\bigvee_{\lambda\in I} \lambda \wedge (\underset{\alpha\in T}{*} f^\alpha(\lambda)) = \underset{\alpha\in T}{*} (\bigvee_{\lambda\in I} \lambda \wedge f^\alpha(\lambda))$$

Proof. For $\forall \alpha \in T$, $f^\alpha(\lambda)$ is a special set. Then $\bigwedge_{\alpha\in T} f^\alpha(\lambda)$, $\bigvee_{\alpha\in T} f^\alpha(\lambda)$ are the same, furthermore $\underset{\alpha\in T}{*} f^\alpha(\lambda)$ is so. Clearly, the proof of theorem 2 we only need consider the equations.

$$\bigvee_{\lambda\in I} \lambda \wedge (\bigvee_{\alpha\in T} f^\alpha(\lambda)) = \bigvee_{\alpha\in T} (\bigvee_{\lambda\in I} \lambda \wedge f^\alpha(\lambda)) \qquad (5)$$

and

$$\bigvee_{\lambda\in I} \lambda \wedge (\bigwedge_{\alpha\in T} f^{\alpha}(\lambda)) = \bigwedge_{\alpha\in T} (\bigvee_{\lambda\in I} \lambda\wedge f^{\alpha}(\lambda)) \tag{4}$$

First by theorem 1 we have

$$\bigvee_{\lambda\in I} \lambda \wedge (\bigwedge_{\alpha\in T} f^{\alpha}(\lambda))$$

$$= \bigwedge_{\lambda\in I} (\lambda \vee (\bigwedge_{\alpha\in T} f^{\alpha}(\lambda))$$

$$= \bigwedge_{\lambda\in I} (\bigwedge_{\alpha\in T} (\lambda\vee f^{\alpha}(\lambda))$$

$$= \bigwedge_{\alpha\in T} (\bigwedge_{\lambda\in I} (\lambda\vee f^{\alpha}(\lambda))$$

$$= \bigwedge_{\alpha\in T} (\bigvee_{\lambda\in I} \lambda\wedge f^{\alpha}(\lambda)) .$$

On the other hand

$$\bigvee_{\lambda\in I} \lambda \wedge (\bigvee_{\alpha\in T} f^{\alpha}(\lambda))$$

$$= \bigvee_{\lambda\in I} \bigvee_{\alpha\in T} (\lambda\wedge f^{\alpha}(\lambda))$$

$$= \bigvee_{\alpha\in T} (\bigvee_{\lambda\in I} \lambda\wedge f^{\alpha}(\lambda)) .$$

Theorem 3. Let H_λ be a nonfuzzy set in U and for $\forall \lambda\in I$ there is

$$H_\lambda(u) = \mathop{*}_{t\in T} A^t_\lambda(a^t) , \tag{6}$$

then

1) $H = \bigcup_{\lambda\in I} \lambda H_\lambda$ is a fuzzy set in U.

2) $H(u) = \mathop{*}_{t\in T} A^t(a^t)$. (7)

where $a^t \in U^t$ is a universe, A^t is a fuzzy set in U^t.

Proof. For $\forall \lambda \in I$, $\forall u \in U$, $\lambda \wedge H_\lambda(u)$ is definite, hence $\bigvee_{\lambda \in I} \lambda H_\lambda(u)$ is also definite, and for $\forall u \in U$, the $\bigvee_{\lambda \in I} \lambda H_\lambda(u) \in I$, this completes the proof 1).

On the other hand

$$\begin{aligned} H(u) &= \bigvee_{\lambda \in I} \lambda \wedge (\underset{t \in T}{*} A_\lambda^t(a^t)) \\ &= \underset{t \in T}{*} (\bigvee_{\lambda \in I} \lambda \wedge A_\lambda^t(a^t)) \\ &= \underset{t \in T}{*} A^t(a^t) . \end{aligned}$$

Remark. From Decomposition theorem we know, that if $H_\lambda = A_\lambda$, then any fuzzy set A can be given in accordance with 1). We call the fuzzy set generated by 1) as "made set". The meaning of theorem 3 is that not only A_λ, $A_{\underset{\cdot}{\lambda}}$ or $H(\lambda)$ can be made a fuzzy set and so the H_λ is any nonfuzzy set, and when condition (6) is satisfied the grade of membership of "made set" keeps the form of original set. This has important value for applycation that makes the fuzzify nonfuzzy concept simple and convenient.

4. EXAMPLES

Example 1. Let X be a cartesion product of universes, $X = X_1 \times \dots \times X_n$ and $A^1, \dots, A^n$ be n fuzzy set in $X_1, \dots, X_n$ is defined as

$$A^1 \times \cdots \times A^n = \bigcup_{\lambda \in I} \lambda (A_\lambda^1 \times \cdots \times A_\lambda^n) .$$

because for $\forall x_i \in X_i$ hold

$$(A_\lambda^1 \times \cdots \times A_\lambda^n)(x_1, \cdots, x_n) = \bigwedge_{i=1}^{n} A_\lambda^i(x_i) ,$$

then by theorem 3 for $\forall x_i \in X_i$, $i = 1, \cdots, n$; there is

$$(A^1 \times \cdots \times A^n)(x_1, \cdots, x_n) = \bigwedge_{i=1}^{n} A^i(x_i) .$$

Example 2. Let f be a mapping from $X = X_1 \times \cdots \times X_n$ to y, the induced mapping f from $F(X_1) \times \cdots \times F(X_n)$ to F(y) is defined as

$$f(A^1 \times \cdots \times A^n) = \bigvee_{\lambda \in I} \lambda f((A^1 \times \cdots \times A^n)_\lambda)$$

the $f(A^1 \times \cdots A^n)$ is a fuzzy set in y. When sets are nonfuzzy we have

$$f((A^1 \times \cdots \times A^n)_\lambda)(y)$$

$$= f(A^1_\lambda \times \cdots \times A^n_\lambda)(y)$$

$$= \bigvee_{y = f(x_1, \cdots, x_n)} \bigwedge_{i=1}^{n} A^i_\lambda(x_i) .$$

by theorem 3 it can be written immediately

$$f(A^1 \times \cdots \times A^n)(y) = \bigvee_{y = f(x_1, \cdots, x_n)} \bigwedge_{i=1}^{n} A^i(x_i) .$$

Example 3. Let s and R be two fuzzy relations on $X \times Y$ and $Y \times Z$, respectively, S_λ , R_λ are ordinary relations. By composition of ordinary relations,

$$(S_\lambda \circ R_\lambda)(x, z) = \bigvee_{y \in Y} S_\lambda(x, y) \wedge R_\lambda(y, z) .$$

if we denote composition of fuzzy relations as a "made set" of composition of their λ-cuts, then

$$(S \circ R)(x, z) = \bigvee_{y \in Y} S(x, y) \wedge R(y, z) .$$

Remark. Example 2 to prove the very important extension principle can get from theorem 3. About other use (e.g. fuzzy algebraic system) we shall discuss on other place.

REFERENCE

[1] Luo Chengzhong "fuzzy set and setembedding". Fuzzy Mathematics 1984 (4). China