FUZZY PARTITIONS OF SETS

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1. Introduction

Let be given any crisp X. Each fuzzy subset $\underset{\sim}{A}$ of X is qualified by means of its membership function $\mu_{\underset{\sim}{A}}: X \to [0,1]$. If it all possible any fuzzy subset $\underset{\sim}{A}$ will be shortly described by mapping $\mu: X \to [0,1]$. Through this paper we shall use the following definitions of complement, union and intersection of fuzzy subsets:

$$\mu_{\underset{\sim}{A}'} = 1 - \mu_{\underset{\sim}{A}} ,$$
$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}} = \max\{\mu_{\underset{\sim}{A}}, \mu_{\underset{\sim}{B}}\} = \mu_{\underset{\sim}{A}} \vee \mu_{\underset{\sim}{B}} ,$$
$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}} = \min\{\mu_{\underset{\sim}{A}}, \mu_{\underset{\sim}{B}}\} = \mu_{\underset{\sim}{A}} \wedge \mu_{\underset{\sim}{B}} .$$

Classical approach to fuzzy subsets is based among other things on the next notions, strict say:

- the fuzzy subset $\mathbb{1}_X: X \to \{1\}$ is called an universum,
- the fuzzy subset $\mathbb{0}_X: X \to \{0\}$ is called an empty set,
- each fuzzy subsets μ and ν which $\mu \wedge \nu = \mathbb{0}_X$ are called a separated sets.

The strict notions do not satisfy the Excluded Middle Law. This fact complicates considerations about fuzzy subsets. A new conception of separatity between fuzzy subsets avioiding this difficulty is presented in [2]. Proposed approach bases on weak notions given below:

- each fuzzy subset $\mu : X \to [0,1]$, which $\mu \leqslant 1-\mu$, is called a W-empty set,
- each fuzzy subset $\mu : X \to [0,1]$, which $\mu \geqslant 1-\mu$, is called a W-universum,
- each fuzzy subsets μ and ν in X, such that $\mu \leqslant 1-\nu$, are called a W-separated sets.

In crisp case the weak notions are equivalent to the strict notions. In general the weak notions are more general than the strict notions. More details about weak notions we can find in cited above paper. Employment of weak notions repairs fundamental differences between fuzzy and crisp theories of probability [3]. In my opinion analogous effect we may obtain for others fuzzy theories. Therefore, the next results for weak notions are presented in this paper.

2. The main problem

Problem of partition of crisp set $A \subset X$ consists in looking for sequence of pairwise separated subsets $\{A_n\}$ such that $\bigcup_n \{A_n\} = A$. Any partition of crisp set A can be defined equivalently as sequence of pairwise separated subsets $\{A_n\}$ which fulfils $\bigcup_n \{A_n\} \subset A$ and $A \setminus \bigcup_n \{A_n\} = A \cap (\bigcup_n \{A_n\}) = \emptyset$. If the sequence $\{A_n\}$ is a partition of crisp universum X then it is called a complete partition.

Let us displace this problem to domain of fuzzy subsets. Kabała and Wrociński propose to accept the following definition based on strict notions.

Definition 2.1: Each sequence of pairwise strict separated fuzzy subsets $\{\mu_n\}$, which $\sup_n \{\mu_n\} = \mathbb{1}_X$ is called a complete partition in strict sense [1].

Furthermore, they prove that any defined above partition contains crisp subsets only. Therefore, this notion is not useful for fuzzy mathematics.

Let us consider analogous definitions based on the weak notions.

Definition 2.2: If finite or infinite sequence of fuzzy subsets $\{\nu_n\}$ fulfils the next properties:

- fuzzy subsets ν_n are pairwise W-separated; (2.1)
- the fuzzy subset $\mu \wedge (1 - \sup_n \{\nu_n\})$ is a W-empty set; (2.2)
- $\sup_n \{\nu_n\} \leqslant \mu$ (2.3)

for fixed subset μ then it is called a partition of μ.

Definition 2.3: Each partition of $\mathbb{1}_X$ is called a complete partition.

Any complete partition can be defined equivalently in other manner because of:

Theorem 2.1: The sequence $\{\nu_n\}$ is a complete partition iff it satisfies (2.1) and:

- the fuzzy subset $\sup_n \{\nu_n\}$ is a W-universum. (2.4)

Proof: Any sequence $\{\nu_n\}$ satisfies (2.3) for $\mu = \mathbb{1}_X$. Furthermore, for $\mu = \mathbb{1}_X$, the condition (2.2) is equivalent to the condition: the fuzzy subset $1 - \sup_n \{\nu_n\}$ is a W-empty set. Since complementation of any W-empty set is a W-universum, the condition (2.2) is equivalent to the condition (2.4). ■

In [3] a complete partition is defined as sequence of fuzzy subsets satisfying (2.1) and (2.4). Can we always find a complete partition which does not contain a crisp subsets only? The answer to this question is presented below.

Theorem 2.2: If the sequence $\{\mu_n\}$ fulfils (2.4) then the sequence $\{\nu_n\}$ defined as follows

$$\nu_n = \begin{cases} \mu_1 & n = 1 \\ \mu_n \wedge (1 - \max_{k<n} \{\nu_k\}) & n > 1 \end{cases} \qquad (2.5)$$

is a complete partition. [3]

3. Properties of partitions

The following theorems describe connections between partitions.

Theorem 3.1: Let be given any pair of fuzzy subsets (μ_1, μ_2) such that $\mu_2 \leqslant \mu_1$. If the sequence $\{\nu_n\}$ is a partition of μ_1 then the sequence $\{\mu_2 \wedge \nu_n\}$ is a partition of μ_2.

Proof: We have: $\mu_2 \wedge \nu_1 \leqslant \nu_1 \leqslant 1 - \nu_k \leqslant 1 - \mu_2 \wedge \nu_k$ for any $k \neq 1$. So, the sequence $\{\mu_2 \wedge \nu_n\}$ satisfies (2.1). Furthermore, $\sup_n \{\mu_2 \wedge \nu_n\} \leqslant \mu_2$. The condition (2.2) holds because we have

$\mu_2 \wedge (1 - \sup_n \{\mu_2 \wedge \nu_n\}) = \mu_2 \wedge (1 - \mu_2 \wedge \sup_n \{\nu_n\}) =$

$$= \mu_2 \wedge ((1 - \mu_2) \vee (1 - \sup_n \{\nu_n\})) = (\mu_2 \wedge (1 - \mu_2)) \vee$$

$$\vee (\mu_2 \wedge (1 - \sup_n \{\nu_n\})) \leqslant [\![\tfrac{1}{2}]\!]_X \vee (\mu_1 \wedge (1 - \sup_n \{\nu_n\})) \leqslant [\![\tfrac{1}{2}]\!]_X$$

where $[\![\frac{1}{2}]\!]_X : X \rightarrow \{\frac{1}{2}\}$. ■

Theorem 3.2: If the sequence $\{\nu_n\}$ is a complete partition, then the sequence $\{\mu \wedge \nu_n\}$ is a partition of μ for any fuzzy subset μ.

Proof: The above thesis immediately follows from the Theorem 3.1 for $\mu_1 = \mathbb{1}_X$. ■

Theorem 3.3: If the sequence $\{\nu_n\}$ is a partition of μ then each sequence $\{\nu'_k\}$ fulfilling the conditions (2.1), (2.3) and $\sup_n \{\nu_n\} \leqslant \sup_k \{\nu'_k\}$ is a partition of μ.

Proof: We have

$$\mu \wedge (1 - \sup_k \{\nu'_k\}) \leqslant \mu \wedge (1 - \sup_n \{\nu_n\}) .$$

So, the sequence $\{\nu'_k\}$ fulfils (2.2), too. ■

We can find a partition of any fuzzy subset μ such that it contains uncrisp subsets. This conclusion follows from the Theorems 2.2 and 3.2. Moreover, we note that we can find a fuzzy partition of any crisp subsets $A \subset X$ described by means of its membership function $\mathbb{1}_A$.

The next theorems present a some examples of partitions.

Lemma 4.1: If the sequence of fuzzy subsets $\{\mu_n\}$ is nondecreasing, then the sequence $\{\nu_n\}$, defined as follows

$$\nu_n = \begin{cases} \mu_1 & n = 1 \\ \mu_n \wedge (1 - \mu_{n-1}) & n > 1 \end{cases} \qquad (3.1)$$

satisfies the next condition

$$1 - \max_{k \leq n} \{\nu_k\} = \max_{k<n} \{\mu_k \wedge (1 - \mu_k)\} \vee (1 - \mu_n) \qquad (3.2)$$

for each $n > 2$. [3]

Lemma 3.2: If the nondecreasing sequence $\{\mu_n\}$ fulfils (2.2) for the fuzzy subset μ then the sequence $\{\nu_n\}$, defined by (3.1), satisfies (2.2) for μ, too.

Proof: By (3.2) we get

$$\mu \wedge (1 - \sup_n \{\nu_n\}) = \mu \wedge (1 - \sup_n \{ \max_{k \leq n} \{\nu_k\}\}) =$$

$$= \mu \wedge \inf_n \{1 - \max_{k \leq n} \{\nu_k\}\} = \mu \wedge \inf_{n \geq 2} \{1 - \max_{k \leq n} \{\nu_k\}\} =$$

$$= \mu \wedge \inf_{n \geq 2} \{ \max_{k<n} \{\mu_k \wedge (1 - \mu_k) \vee (1 - \mu_n)\} \leq$$

$$\leq \mu \wedge \inf_{n \geq 2} \{[\![\tfrac{1}{2}]\!]_X \vee (1 - \mu_n)\} = (\mu \wedge [\![\tfrac{1}{2}]\!]_X) \vee (\mu \wedge \inf_{n \geq 2} \{1 - \mu_n\}) \leq$$

$$\leq [\![\tfrac{1}{2}]\!]_X \vee (\mu \wedge (1 - \sup_{n \geq 2} \{\mu_n\})) = [\![\tfrac{1}{2}]\!]_X \vee (\mu \wedge (1 - \sup_n \{\mu_n\})) \leq$$

$$\leq [\![\tfrac{1}{2}]\!]_X .$$

So, the condition (2.2) holds for $\{\nu_n\}$, too. ■

Lemma 3.3: If the sequence $\{\mu_n\}$ fulfils (2.2) for the fuzzy subset μ then the sequence $\{\nu_n\}$ given by

$$\nu_n = \begin{cases} \mu_1 & n = 1 \\ \mu_n \wedge (1 - \max_{k<n} \{\mu_k\}) & n > 1 \end{cases} \qquad (3.3)$$

satisfies (2.2) for μ.

Proof: Let us define the sequence

$$\psi_n = \begin{cases} 0_X & n = 1 \\ \max\limits_{k<n} \{\mu_n\} & n > 1 \end{cases}$$

The sequence $\{\psi_n\}$ is nondecreasing. It is easy to check that $\{\psi_n\}$ fulfils (2.2) for μ. Moreover, we have $\nu_n = \mu_n \wedge \wedge (1 - \psi_n)$. Hence the Lemma 3.2 implies

$$\mu \wedge (1 - \sup_n \{\nu_n\}) = \mu \wedge (1 - \sup_n \{\max_{m\leqslant n} \{\nu_m\}\}) =$$

$$= \mu \wedge (1 - \sup_n \{\max_{m\leqslant n} \{\mu_m \wedge (1 - \psi_m)\}\}) \leqslant$$

$$\leqslant \mu \wedge (1 - \sup_n \{\max_{m\leqslant n} \{\mu_m \wedge (1 - \psi_n)\}\}) = \mu \wedge (1 - \sup_n \{\psi_{n+1} \wedge (1 - \psi_n)\}) \leqslant$$

$$\leqslant [\![\tfrac{1}{2}]\!]_X$$

Thus the condition (2.2) holds.■

Theorem 3.4: If the sequence $\{\mu_n\}$ fulfils (2.2) and (2.3) for the fuzzy subset μ then the sequence $\{\nu_n\}$, defined by (3.3) is a partition of μ.

Proof: Since $\nu_n \leqslant \mu_n$, the sequence $\{\nu_n\}$ satisfies (2.3) for μ. Furthermore, the fuzzy subsets ν_n are pairwise W-separated because we have

$$\nu_n = \mu_n \wedge (1 - \max_{l<n} \{\mu_l\}) \leqslant 1 - \max_{l<n} \{\mu_l\} \leqslant 1 - \mu_k \leqslant 1 - \nu_k$$

for any pair (n,k) such that $n > k$. The Lemma 3.3 puts an end to the proof.■

Theorem 3.5: If the nondecreasing sequence $\{\mu_n\}$ fulfils (2.2) and (2.3) for the fuzzy subset μ then the sequence $\{\nu_n\}$, defined by (3.4), is a partition of μ .

Proof: If the sequence $\{\mu_n\}$ is nondecreasing, then the sequences defined by (3.3) or (3.4) are identical. ■

Theorem 3.6: If the sequence $\{\mu_n\}$ fulfils (2.2) and (2.3) for the fuzzy subset μ then the sequence $\{\nu_n\}$, defined by (2.5), is a partition of μ .

Proof: From the Theorem 2.2 we obtain that $\{\nu_n\}$ satisfies (2.1). Since $\nu_n \leq \mu_n$, the condition (2.3) holds for $\{\nu_n\}$ and μ , too. Moreover, the last inequality implies

$$\sup_n \{\nu_n\} = \mu_1 \vee \sup_{n\geq 2}\{\mu_n \wedge (1 - \max_{k<n}\{\nu_k\})\} \geq$$

$$\geq \mu_1 \vee \sup_{n\geq 2}\{\mu_n \wedge (1 - \max_{k<n}\{\mu_k\})\}$$

This fact together with the Theorems 3.3 and 3.5 proves the thesis. ■

Theorem 3.7: Let $\{\mu_k\}_{k=1}^{n}$ be finite nondecreasing sequence of fuzzy subsets and $n \geq 2$. Then the sequence $\{\mu_{k+1} \wedge (1 - \mu_k)\}_{k=1}^{n-1}$ is a partition of $\mu_n \wedge (1 - \mu_1)$.

Proof: The Theorem 3.5 implies that the fuzzy subsets $\mu_{k+1} \wedge (1 - \mu_k)$ are pairwise W-separated. Since

$$\sup_k \{\mu_{k+1} \wedge (1 - \mu_k)\} = \max_{k<n}\{\mu_{k+1} \wedge (1 - \mu_k)\} \leq \mu_n \wedge (1 - \mu_1) ,$$

considered sequence fulfils (2.3). By the Lemma 3.1 we get

$$\mu_n \wedge (1 - \mu_1) \wedge (1 - \sup_k\{\mu_{k+1} \wedge (1 - \mu_k)\}) =$$

$$= \mu_n \wedge (1 - \mu_1) \wedge (1 - \max_{k<n}\{\mu_{k+1} \wedge (1 - \mu_k)\}) =$$

$$= \mu_n \wedge (1 - \mu_1 \vee \max_{k<n}\{\mu_{k+1} \wedge (1 - \mu_k)\}) =$$

$$= \mu_n \wedge \left(\max_{k<n} \{\mu_k \wedge (1 - \mu_k)\} \vee (1 - \mu_n)\right) \leqslant$$

$$\leqslant \mu_n \wedge \left([\![\tfrac{1}{2}]\!]_X \vee (1 - \mu_n)\right) = \left(\mu_n \wedge [\![\tfrac{1}{2}]\!]_X\right) \vee \left(\mu_n \wedge (1 - \mu_n)\right) \leqslant$$

$$\leqslant [\![\tfrac{1}{2}]\!]_X .$$

Thus [illegible].2 holds, too. ■

All above results can be useful for applications of the Bayes Formula given for fuzzy events [3].

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