# THE DEFINITION OF DISTANCE AND DIAMETER IN FUZZY SET THEORY

by

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## 1. INTRODUCTION

In any metric space (S,d) it is possible to define the distance between two subsets X and Y of S by setting $\delta(X,Y)=0$ if $X=\emptyset$ or $Y=\emptyset$ and

(1) $\delta(X,Y)=\inf\{d(x,y)/x\in X, y\in Y\}$ otherwise.

The distance between a point x and a set X is defined by setting $\delta(x,X)=\delta(\{x\},X)$.

This allows, for instance, to characterize the non-empty closed sets as the sets X for which $x\in X$ if and only if $\delta(x,X)=0$.

Another foundamental concept is that of diameter $\Delta(X)$ of a set. One defines it by setting $\Delta(X)=0$ if $X=\emptyset$ and

(2) $\Delta(X)=\sup\{d(x,y)/x\in X, y\in X\}$ otherwise.

In this paper our aim is to define analogue concepts for the fuzzy sets. So we define the distance between two fuzzy sets and, hence, between a fuzzy point and a fuzzy set.

We call closed a fuzzy set containing all the fuzzy points that have distance from it equals to zero, and we show that the complements of closed sets determine a fuzzy topology, the fuzzy topology of lower semi-continuous functions.

Also, we define the diameter of a fuzzy set. This will allow to characterize the fuzzy points as the fuzzy sets with diameter equals to zero.

## 2. PREREQUISITES AND DEFINITIONS

Let X be a set and R the set of real numbers. We say fuzzy subset of X or, more simply, fuzzy set [8] a function $f:X\rightarrow[0,1]$ where $[0,1]$ denotes the set $\{\alpha\in R/0\leq\alpha\leq 1\}$.

We denote by F(X) the class of the fuzzy subsets of X.

If $f,g\in F(X)$ then we set $f\leq g$ iff $f(x)\leq g(x)$ for any $x\in X$.

Moreover $-f$, <u>the complement of f</u> is the fuzzy subset of X defined by setting $(-f)(x)=1-f(x)$ for any $x\in X$. If $(f_i)_{i\in I}$ is a family of fuzzy subsets of X then $\bigvee_{i\in I} f_i$ and $\bigwedge_{i\in I} f_i$ are the fuzzy subsets of X defined by setting

$(\bigvee_{i\in I} f_i)(x) = \sup_{i\in I}\{f_i(x)\}$ and

$(\bigwedge_{i\in I} f_i)(x) = \inf_{i\in I}\{f_i(x)\}$ for any $x\in X$.

We denote by $f_0$ and $f_1$ the fuzzy sets for which $f_0(x)=0$ and $f_1(x)=1$ for any $x\in X$.

Moreover, if $\alpha\in[0,1]$, we call <u>$\alpha$-cut</u> of a fuzzy set f the subset $C_f^\alpha=\{x\in X/f(x)\geq\alpha\}$.

A fuzzy set f is called <u>crisp</u> if $f(x)\in\{0,1\}$ for any $x\in X$. The fuzzy sets crisp can be interpreted as characteristic functions of subsets of X and, hence, they can be identified with these subsets.

For any $a\in X$ and $\alpha\in(0,1]=\{x\in R/0<x\leq 1\}$ the fuzzy set $f_a^\alpha$, defined by setting $f_a^\alpha(x)=0$ if $x\neq a$ and $f_a^\alpha(x)=\alpha$ if $x=a$, is called <u>fuzzy point</u> ([7], [3], [4]).

We say that the fuzzy point $f_a^\alpha$ <u>belongs</u> to the fuzzy set f, $f_a^\alpha\in f$, if $f_a^\alpha\leq f$ that is if $f(a)\geq\alpha$.

We can now define the concept of fuzzy topological space ( see references). To this aim we give the following definitions.

DEFINITION 1. A class $\mathcal{T}$ of fuzzy subsets of X constitutes a <u>fuzzy topology</u> if the following conditions are verified:

a) $f_0,\ f_1\in\mathcal{T}$

b) if $f,g\in\mathcal{T}$ then $f\wedge g\in\mathcal{T}$

c) $\bigvee_{i\in I} f_i\in\mathcal{T}$ for any family $(f_i)_{i\in I}$ of elements in $\mathcal{T}$.

The pair $(X,\mathcal{T})$ is named <u>fuzzy topological space</u>; the elements of $\mathcal{T}$ are named <u>open</u>, the complements of these elements are named <u>closed</u>.

The following definition is dual of Definition 1.

DEFINITION 2. A class C F(X) is a <u>system of closed fuzzy subsets of X</u> if the following conditions are verified:

a) $f_0, f_1 \in C$

b) if $f, g \in C$ then $f \vee g \in C$

c) $\bigwedge_{i \in I} f_i \in C$ for any family $(f_i)_{i \in I}$ of elements of C.

Obviously, the class of complements of a system of closed fuzzy set is a fuzzy topology and the class of complements of a fuzzy topology constitutes a system of closed fuzzy sets.

## 3. DISTANCE BETWEEN TWO FUZZY SETS

Let (S,d) be a metric space. We define a <u>distance</u> between two fuzzy subsets f,g of S in the following way:

(3) $d(f,g)=\int_0^1 \delta(C_f^\alpha, C_g^\alpha)d\alpha$

Note that if $\beta \geq \alpha$ then $C_f^\beta = \{x \in S/f(x) \geq \beta\} \subseteq C_f^\alpha = \{x \in S/f(x) \geq \alpha\}$ and, hence, $\delta(C_f^\beta, C_g^\beta) \geq \delta(C_f^\alpha, C_g^\alpha)$. This proves that $\delta(C_f^\alpha, C_g^\alpha)$ is an increasing function of $\alpha$ and, hence, that the distance between two fuzzy sets is defined for any $f,g \in F(S)$, even if it is finite or infinite. An example of a pair of fuzzy sets with infinite distance is the following.

Let (S,d) be the set of real numbers with the usual distance, and consider $f_0^1$ and f, where f is the fuzzy set for which $f(x)= x/x+1$ ; then $d(f_0^1, f)$ is equal to $\infty$.

If in f and g there are two crisp points,that is if there exist x,y in S for which $f_x^1$ and $f_y^1$ belong rispectively to f and g, then, being any contribution $\delta(C_f^\alpha, C_g^\alpha) \leq d(x,y)$, the integral in (3) assumes a finite value.

If f and g are the characteristic functions of two subsets X and Y of S then $C_f^\alpha = X$ and $C_g^\alpha = Y$ for every $\alpha > 0$, hence $d(f,g) = \int_0^1 \delta(X,Y)d\alpha = 1 \cdot \delta(X,Y) = \delta(X,Y)$. Then (3) generalizes the classical definition of distance between two subsets of a metric space.

Obviously the distance between a fuzzy point $f_x^\alpha$ and a fuzzy set g is $\int_0^\alpha \delta(x, C_g^\beta)d\beta$ . Moreover the distance between two fuzzy points $f_b^\beta$ and $f_c^\gamma$ is equal to $\int_0^{\beta \wedge \gamma} \delta(\{b\},\{c\})d\alpha$ and therefore

(4) $d(f_b^{\beta}, f_c^{\gamma}) = [\gamma \wedge \beta] \cdot d(b,c)$ .

This proves that, for the fuzzy points crisp , the distance defined by (3) coincides with the usual one between points.

It is interesting to examine the case that f and g assume values in a finite subset $\{\gamma_0, \ldots, \gamma_n\}$ of $[0,1]$. Then, if $0=\gamma_0<\gamma_1<\ldots<\gamma_n=1$ we have

(5) $d(f,g) = \sum_{i=1}^{n} \delta(C_f^{\gamma_i}, C_g^{\gamma_i}) \cdot (\gamma_i - \gamma_{i-1})$;

if, for $i=1,\ldots,n$, $\gamma_i - \gamma_{i-1} = 1/n$

(6) $d(f,g) = 1/n \cdot (\sum_{i=1}^{n} \delta(C_f^{\gamma_i}, C_g^{\gamma_i}))$.

In general, we can also utilize Formulas (5) and (6) to compute a suitable approximation of the distance between two fuzzy subsets.

We can give a definition of closure for fuzzy sets:

DEFINITION 3. A fuzzy set f is <u>metrically closed</u> if either $f=f_0$ or, for every fuzzy point $f_x^{\alpha}$, $f_x^{\alpha} \in f$ iff $d(f_x^{\alpha}, f)=0$. We denote by C the class of the metrically closed fuzzy sets.

PROPOSITION 1. The set C is a system of closed fuzzy subsets of X. Equivalently, the set $\mathcal{T}$ of the relative complements defines a fuzzy topology.

PROOF. It is obvious that $f_0$ and $f_1$ are elements of C. Let $f \in C$ and $g \in C$, and let $f_x^{\alpha}$ a fuzzy point. If $f_x^{\alpha} \in f \vee g$ it is obvious that $d(f_x^{\alpha}, f \vee g)=0$. Conversely, suppose that $d(f_x^{\alpha}, f \vee g)=0$, then $\delta(x, C_{f\vee g}^{\beta})=0$ for every $\beta<\alpha$. Suppose, by absurd that $f_x^{\alpha} \notin f \vee g$, then $f(x)<\alpha$ and $g(x)<\alpha$ , i.e. $f_x^{\alpha} \notin f$ and $f_x^{\alpha} \notin g$. This implies that $d(f_x^{\alpha}, f)>0$ and $d(f_x^{\alpha}, g)>0$ and therefore that $\delta(x, C_f^{\gamma})>0$ and $\delta(x, C_g^{\gamma})>0$ for a suitable $\gamma<\alpha$. It follows that $\delta(x, C_f \cup C_g)>0$ and, since $C_{f\vee g}^{\gamma} \subseteq C_f^{\gamma} \cup C_g^{\gamma}$, $\delta(x, C_{f\vee g}^{\gamma}) \geq \delta(x, C_f^{\gamma} \cup C_g^{\gamma})>0$, an absurd. This prove that $f_x^{\alpha} \in f \vee g$ and therefore that $f \vee g \in C$.

Let $(f_i)_{i \in I}$ be a family of elements of C and set $f = \bigwedge_{i\in I} f_i$: we have to prove that $f \in C$. If $f_x^{\alpha} \in f$ it is obvious that $d(f_x^{\alpha}, f)=0$. Assume that $d(f_x^{\alpha}, f)=0$, then $\delta(x\ C_f^{\beta})=0$ for every $\beta<\alpha$. If, by absurd, $\alpha>f(x)$, then $\alpha>f_j(x)$, and therefore $f_x^{\alpha} \notin f_j$, for a suitable

$j \in I$. Thus $\int_0^{\alpha} \delta(x, C_{f_j}^{\beta}) d\beta > 0$ and there exists $\gamma < \alpha$ such that $\delta(x, C_{f_j}^{\gamma}) > 0$. Since $C_f^{\gamma} \subseteq C_{f_j}^{\gamma}$, we have also that $\delta(x, C_f^{\gamma}) \geq (x, C_{f_j}^{\gamma}) > 0$, an absurd. Thus we have proved that $\alpha \leq f(x)$ and therefore that $f_x^{\alpha} \in f$. This complete the proof.

Now we show that the above defined fuzzy topology $\mathcal{C}$ coincides with the <u>natural fuzzy topology</u> defined in [2].

PROPOSITION 2. C is the class of the upper semicontinuous functions from S to $[0,1]$. It follows that $\mathcal{C}$ is the class of the lower semicontinuous functions.

PROOF. Let $f \in C$, then, to prove that f is upper semicontinuous, it suffices to prove that $\{x \in S / f(x) < \alpha\}$ is open for every $\alpha \in [0,1]$. Equivalently, we can prove that $C_f^{\alpha}$ is closed. Let $x \in S$ and $\delta(x, C_f^{\alpha}) = 0$, then $\delta(x, C_f^{\beta}) = 0$ for every $\beta < \alpha$ and therefore $d(f_x^{\alpha}, f) = \int_0^{\alpha} \delta(x, C_f^{\beta}) d\beta = 0$. Thus $f_x^{\alpha} \in f$ and $x \in C_f^{\alpha}$. This proves that $C_f^{\alpha}$ is closed.

Conversely, suppose f upper semicontinuous or, equivalently, that $C_f^{\alpha}$ is closed for every $\alpha \in [0,1]$. Moreover, suppose that $d(f_x^{\alpha}, f) = \int_0^{\alpha} \delta(x, C_f^{\beta}) d\beta = 0$, then $\delta(x, C_f^{\beta}) = 0$ for every $\beta < \alpha$. This implies that $x \in C_f^{\beta}$ and therefore that $f(x) \geq \beta$ for every $\beta < \alpha$. In conclusion $f(x) \geq \alpha$ and $f_x^{\alpha} \in f$. This proves that $f \in C$.

## 4. DIAMETER OF A FUZZY SET

Let f be a fuzzy subset of S, then we set

(7) $\Delta(f) = \sup\{ d(x,y) / x$ and y are fuzzy points of $f\}$.

The number $\Delta(f)$ may be either finite or infinite, we call it the <u>diameter of the fuzzy set f</u>.

If $\Delta(f)<\infty$ then f is **called** **bounded**.

Being $\delta(f_x^\alpha, f_y^\beta) = d(x,y)\cdot(\alpha\wedge\beta)$, it is obvious that

(8) $\Delta(f) = \sup\{d(x,y)\cdot[f(x)\wedge f(y)] / x,y\in S\}$.

PROPOSITION 3. If f is crisp then the definition of diameter is the classical one. Moreover if $f\leq g$ then $\Delta(f)\leq\Delta(g)$.

PROOF. If f is the characteristic function of the set X then $\Delta(f) = \sup\{d(x,y).[f(x)\wedge f(y)] / f(x)\neq 0\ f(y)\neq 0\} = \sup\{d(x,y)/ x\in X,\ y\in X\}$.

Suppose that $f\leq g$, then $d(x,y).[f(x)\wedge f(y)]\leq d(x,y)\cdot[g(x)\wedge g(y)]$ and $\Delta(f)\leq\Delta(g)$.

PROPOSITION 4. The diameter of a fuzzy set $f\neq f_0$ is equal to zero iff f is a fuzzy point.

PROOF. It is obvious that the diameter of a fuzzy point is zero. Conversely, suppose that f is a fuzzy set for which $\Delta(f)=0$. Then, by (8), $d(x,y)\cdot[f(x)\wedge f(y)]=0$ for every $x,y\in S$. By hypothesis, there exists $a\in S$ for which $f(a)\neq 0$ and, if $y\neq a$, since $d(a,y)\neq 0$ then $f(a)\wedge f(y)=0$. This proves that $f(y)=0$ for every $y\neq a$ and therefore that f is a fuzzy point.

PROPOSITION 5. For any $f\in F(S)$ and $\alpha\in(0,1]$

(9) $\Delta(C_f^\alpha)\leq\Delta(f)/\alpha$

Then every $\alpha$-cut of a bounded fuzzy set is bounded while the converse falls.

PROOF. If $x,y\in C_f^\alpha$, i.e. $f(x)\geq\alpha, f(y)\geq\alpha$, then $d(x,y)\cdot[f(x)\wedge f(y)]\geq d(x,y)\cdot\alpha$. This proves that $\Delta(f)\geq\alpha\cdot d(x,y)$ or, equivalently, $d(x,y)\leq\Delta(f)/\alpha$.

To prove that there exists a fuzzy set f such that $\Delta(f)=\infty$ and $\Delta(C_f^\alpha)<\infty$ for any $\alpha\in[0,1]$, let S be the positive real numbers set and **define** $f:S\to[0,1]$ by setting $f(x)=1/(\sqrt{x}+1)$. Now $\Delta(f)\geq d(0,x)\cdot(f(0)\wedge f(x)) = x/(\sqrt{x}+1)$ for any $x\in S$. Then $\Delta(f)=\infty$ while it is obvious that every cut of f is bounded.

Proposition 5 shows that our definition of bounded fuzzy set is different from Kaufmann's definition [6].

In metric space theory one proves that a subset is bounded if and only if it is contained in a suitable circle. In order to obtain a similar result for fuzzy subsets we give the following definition.

DEFINITION 4. We call <u>f-circle with center</u> $f_c^\gamma$ <u>and</u> <u>radius</u> r, the fuzzy set $C(f_c^\gamma, r)$ such that, for any fuzzy point $f_b^\beta$, $f_b^\beta \in C(f_c^\gamma, r)$ iff $d(f_b^\beta, f_c^\gamma) \le r$ and $\beta \le \gamma$.

PROPOSITION 6. The f-circle $C(f_c^\gamma, r)$ is the fuzzy set defined by

$$(10) \quad f(z) = \begin{cases} \gamma & \text{if } d(z,c) \le r/\gamma \\ r/d(z,c) & \text{otherwise.} \end{cases}$$

Moreover the diameter of $C(f_c^\gamma, r)$ is not greater than 2r.

PROOF. By definition $f = \bigvee \{ f_x^\beta / \beta \le \gamma, x \in S \text{ and } d(f_x^\beta, f_c^\gamma) \le r \}$, then $f(z) = \bigvee \{ f_z^\beta / \beta \le \gamma \text{ and } d(f_z^\beta, f_c^\gamma) \le r \} = \bigvee \{ \beta / \beta \le \gamma \text{ and } \beta \cdot d(x,c) \le r \}$. This proves (10).

To show that $\Delta(f) \le 2r$ observe that, for every pair of fuzzy points $f_b^\beta, f_{b'}^{\beta'}$ with $\beta \le \gamma$ and $\beta' \le \gamma$, the following triangular inequality holds:

$$(11) \quad d(f_b^\beta, f_{b'}^{\beta'}) \le d(f_b^\beta, f_c^\gamma) + d(f_c^\gamma, f_{b'}^{\beta'}).$$

In fact, by $d(b,b') \le d(b,c) + d(c,b')$, we have $(\beta \wedge \beta') \cdot d(b,b') \le (\beta \wedge \beta') \cdot d(b,c) + (\beta \wedge \beta') \cdot d(c,b') \le \beta \cdot d(b,c) + \beta' \cdot d(c,b') = (\beta \wedge \gamma) \cdot d(b,c) + (\beta' \wedge \gamma) \cdot d(c,b') = d(f_b^\beta, f_c^\gamma) + d(f_c^\gamma, f_{b'}^{\beta'})$. But $(\beta \wedge \beta') \cdot d(b,b') = d(f_b^\beta, f_{b'}^{\beta'})$ and then (11) is proved.

From this it follows that $\Delta(f) \le 2r$.

PROPOSITION 7. Let f be a bounded fuzzy set, $\gamma = \sup \{ f(x) \}$ and $c \in S$ a point such that $f(c) > 0$. Then f is contained in the f-circle $C(f_c^\gamma, \Delta(f)/f(c))$. It follows that a fuzzy set f is bounded if and only if it is contained in an f-circle.

PROOF. Let $r = \Delta(f)/f(c)$ and denote by g the f-circle $C(f_c^\gamma, r)$. If $d(z,c) \le r/\gamma$ then $g(z) = \gamma = \sup \{ f(x) \}$ and therefore $g(z) \ge f(z)$. If $d(z,c) > r/\gamma$ then $g(z) = r/d(z,c)$. Since $f(c) \cdot f(z) \le f(c) \wedge f(z)$, we have also that $d(c,z) \cdot f(c) \cdot f(z) \le d(c,z) \cdot (f(c) \wedge f(z)) \le \Delta(f)$. This proves that $f(z) \le r/d(z,c) = g(z)$.

Finally, observe that, if $(S,d)$ is the euclidean plane, then the diameter of an f-circle $C(f_c^\gamma, r)$ is just $2r$. Indeed, let $z$ and $z'$ two points collinear with $c$ such that $d(z,c)=d(z',c)=r/\gamma$. Then $d(z,z')=2r/\gamma$ and $d(f_z^\gamma, f_{z'}^\gamma)=\gamma\cdot d(z,z')=2r$. Since $f_z^\gamma$ and $f_{z'}^\gamma$ are fuzzy points of the f-circle $C(f_c^\gamma, r)$, this proves that the relative diameter is $2r$.

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