Two Classes of Separation Axioms on Topological Molecular Lattice

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ABSTRACT

In this paper the author, on the basis of the work in [4], [5] and [6], proposes two classes of separation axioms in topological molecular lattice, which are generalizations of separation axioms in general topological spaces. In particular, the author has proved that under one class of these separation axioms the convergence net in Hausdorff Space has unique limit.

1. Introduction

In this paper two classes of separation axioms are introduced in topological molecular lattice, which is first proposed by Wang Guojun in paper [2] , one class which is defined by use of the open-neighborhood is one-sided separation axioms, the other which is defined by use of the open- and far-neighborhood is two-sided separation axioms, these two classes of separation axioms both are the generalizations of separation axioms in general topological spaces. The thoughts of two-sided originated from Wang Peizhuang's paper [3], this paper also discussed the convergences of two kinds of molecular net, one is relative to the structure of open-neighborhood, the other is relative to the structure of open- and far-neighborhood.

2. One class of one-sided separation axioms

Lattice L which are used in this paper always is the Topological Molecular Lattice (TML) proposed by Wang Guojun in [2], so the symbols and results please read the paper [2].

Definition 1.1 Let $(L,\mathcal{J})$ be a TML, for any $A\in L$, $U \in L$ is called a neighborhood of A, if there exists $V\in\mathcal{J}$ suth that $A\leqslant V\leqslant U$. The family consisting of all the neighborhoods of A is denoted by $\rho(A)$, if U is neighborhood of A and $U\in\mathcal{J}$, then U is called an open-neighborhood of A, the family consisting of all the open-neighborhoods of A is denoted by $\rho_0(A)$.

For convenience, we define one symbol.

Definition 1.2 Let $(L,\mathcal{J})$ be a TML and for any $A \in L$, the intersection of all open sets containing A is denoted by $O(A)=\wedge\{B: B\in \rho_0(A)\}$.

Theorem 1.1 Let $L(\pi)$ be a TML, $A,B \in I$, then

(1) $A \leq O(A)$;

(2) If $A \leq B$, then $O(A) \leq O(B)$;

(3) $O(A \vee B) \geq O(A) \vee O(B)$;

(4) $O(A \wedge B) \leq O(A) \wedge O(B)$;

(5) If A is open set or closed set, then $O(O(A))=O(A)$.

The proof is straightforward.

Definition 1.2 A TML $L(\pi)$ is called a T_{-1}-space iff, for any molecule a , $b \in \pi$ and $a < b$, there exists $U \in \mathcal{J}_0(a)$ suth that $b \not\leq U$.

Theorem 1.2 A TML $L(\pi)$ is a T_{-1}-space iff, for any $a \in \pi$, a is a component of $O(a)$.

Proof: Necessity. Let $a \in \pi$, from Definition 1.1, we know that $a \leq O(a)$, if a is not a component of $O(a)$, we can suppose that $b \in \pi$ which is a component of $O(a)$, and $a < b \leq O(a)$. Because $L(\pi)$ is T_{-1}-space, for $a < b$, there exists $V \in \mathcal{J}_0(a)$ suth that $b \not\leq V$, that is, $b \not\leq \wedge \{ V;\ V \in \mathcal{J}_0(a) \} = O(a)$, this contradicts the fact that b is a component of $O(a)$.

Sufficiency. For any $a, b \in \pi$ and $a < b$, because a is a component of $O(a)$, we have $b \not\leq O(a)$, so that there exists $V \in \mathcal{J}_0(a)$ suth that $b \not\leq V$, from the arbitrariness of a,b ,we know that $L(\pi)$ is a T_{-1}-space.

Definition 1.4 A TML $L(\pi)$ is called a T_0-space iff, for any a, $b \in \pi$ and $a \neq b$, either there exists $U \in \mathcal{J}_0(a)$ suth that $b \not\leq U$, or there exists $V \in \mathcal{J}_0(b)$ suth that $a \not\leq V$.

It is clear that $T_0 \Rightarrow T_{-1}$.

Theorem 1.3 A TML $L(\pi)$ is T_0-space iff, for any $a, b \in \pi$ and $a \neq b$, either $a \not\leq O(b)$ or $b \not\leq O(a)$.

Proof: Let $L(\pi)$ be a T_0-space, a, $b \in \pi$, $a \neq b$, from the Definition 1.4 , we may assume that there exists $U \in \rho_0(a)$ suth that $b \nleq U$, hence $b \nleq \wedge \{ B; B \in \rho_0(a) \} = O(a)$.

On the other hand, if $a \nleq b$, we may assume that $a \nleq O(b)$, then we have $V \in \rho_0(b)$ suth that $a \nleq V$, i.e., $L(\pi)$ is a T_0-space.

Definition 1.5 A TML $L(\pi)$ is called a T_1-space iff, for any $a \ngeq b$, there exists $U \in \rho_0(a)$ suth that $b \nleq U$.

It is clear that $T_1 \Rightarrow T_0$.

Theorem 1.4 A TML $L(\pi)$ is T_1-space iff , for any $a \in \pi$, we have $O(a) = a$.

Proof: Necessity. Let $L(\pi)$ be a T_1-space, from Theorem 1.2 we know that if $a \in \pi$, then a is a component of $O(a)$. If $O(a)$ has another component b suth that $a \wedge b = o$, from the hypothesis, we know that there exists $V \in \rho_0(a)$ suth that $b \nleq V$, that is, $b \nleq O(a)$, this contradicts the fact that b is a component of $O(a)$, so $O(a)$ has one component, that is, $O(a) = a$ for any $a \in \pi$.

Sufficiency. If for any $a \in \pi$, we always have $a = O(a)$, then if $a, b \in \pi$, $a \angle b$, it follows that $b > O(a)$, that means there exists $U \in \rho_0(a)$ suth that $U \ngeq b$; if $a \wedge b = o$, then $a \nleq O(b)$ and $b \nleq O(a)$. so there exist $V \in \rho_0(b)$, $U \in \rho_0(a)$ suth that $a \nleq V$, $b \nleq U$, that is, $L(\pi)$ is T_1-space.

Definition 1.6 Let $L(\pi)$ be a TML, $A \in L$, if A only has finite components, then A is called a finite set ; if A only has countable components, then A is called a countable set; if A has infinite components, then is infinite set.

Theorem 1.5 Let $(L, \mathcal{T})$ be a T_1-space, if $\mathcal{T}$ has finite base, then π is finite.

Proof: Let $\mathcal{J}_0 = \{V_1, \ldots, V_n\}$ be a finite base for T_1-space $L(\pi)$, for any molecule $a \in \pi$, there exist $V_i \in \mathcal{J}_0$ suth that $a \leq V_i$. Let $V_{i_1}, \ldots, V_{i_k}$ be all elements of $\mathcal{J}_0$ which contain a, let $V = \wedge\{V_{i_j};\ j=1,\ldots,k\}$, then $V \in \mathcal{J}$ and $a \leq V$. Obviously, for any open set U, if $a \leq U$, then $V \leq U$, by using Theorem 1.4 , we have V=a, so a must be the union of some elements of base $\mathcal{J}_0$, that means, for any $a \in \pi$, we have $a \in \mathcal{J}_0$, if $a, b \in \pi$ and $a \neq b$, we also have $a, b \in \mathcal{J}_0$, $a \neq b$, because $\mathcal{J}_0$ is finite, so π is finite, and our theorem is proved.

Definition 1.7 A TML $L(\pi)$ is called a T_2-space iff, for any $a, b \in \pi$ and $a \wedge b = o$, there exist $U \in \rho_0(a)$, $V \in \rho_0(b)$ such that $U \wedge V = o$.

It is clear that $T_2 + T_{-1} \Rightarrow T_1$.

Theorem 1.6 If a TML $L(\pi)$ is a T_2-space then for any $a \in \pi$, we have $O(a) \in \pi$.

Proof: Let $L(\pi)$ be a T_2-space, suppose that $a \in \pi$, if O(a) is not a molecule, then there exist more than two components at least in it, we may assume b, c both are O(a)' components, and $b \geqslant a$. From $b \dot{\wedge} c = o$ we obtain $a \wedge c = o$, so we have $U \in \rho_0(a)$ and $V \in \rho_0(c)$ suth that $U \wedge V = o$, that is, $c \nleq O(a)$. This contradicts the fact that c is a component of O(a), so $O(a) \in \pi$, this completes the proof of the theorem.

Now we give the definitions of T_3- and T_4-space as follow.

Definition 1.8 A TML $L(\pi)$ is called a T_3-space iff, for any $a \in \pi$, $A \in L$, A is closed set and $a \wedge A = o$, there exist $U \in \rho_0(a)$, $V \in \rho_0(A)$ such that $U \wedge V = o$.

Definition 1.9 A TML $L(\pi)$ is called a T_4-space iff, for any $A, B \in L$, $A \wedge B = o$ and A,B both are closed sets, there exist

$U \in \rho_o(A)$, $V \in \rho_o(B)$ suth that $U \wedge V = o$.

Theorem 1.7 Let $L(\pi)$ be a TML, if every molecule is closed set, then T_3-(and T_4-)space also is T_2-space.

The proof, being straightforward, is omitted.

Similar with the general topology, we have some conclusions as follow (the concepts of sublattice and topological sublattice in TML can be found in paper [2] written by Wang Guojun).

Theorem 1.8 If a TML $(L(\pi),\mathcal{T})$ is T_i $(i=-1,o,1,2,3)$-space, then its sublattice $(L|E, E\wedge\mathcal{T})$ is also T_i $(i=-1,o,1,2,3)$-space.

2. The Convergence of Molecular Net

In this part we shall discuss the convergency of molecular net about open-neighborhood.

Definition 2.1 Let $(D,\geqslant)$ be a directed set and $L(\pi)$ a molecular lattice, the function $S: D \to \pi$ is called a molecular net in L, and is denoted by $S=\{S(n), n\in D\}$; S is said to be in A iff, for each $n\in D$, $S(n)\leq A$; Let $L(\pi)$ be a TML, a net S in L is said to converge to a, or a is said a limit of S iff S eventually belongs to each $U \in \rho_o(a)$, and is denoted by $S\to a$. The set of all limits of the net S is denoted by LimS. a is said to be a cluster of a net S iff, for each $U\in \rho(a)$, S frequently belongs to U.

The convergence defined as above is relative to open-neighborhood, the convergence which was discussed in paper [2] is relative to far-neighborhood, obviously,both convergences are distinct.

Theorem 2.1 Let $L(\pi)$ be a TML, for any $a\in\pi$, if a is a limit of net S, then for any $b\in\pi$, $b\geqslant a$, b is also the limit of S; similarly, if a is a cluster of net S, $b\in\pi$, $b\geqslant a$, then b is also the cluster of S.

Proof: We shall now prove the first part of the Theorem, the proof of the last part is similar. Suppose that $S \rightarrow a$, $a \in \pi$, $b \in \pi$, $b > a$, for any $U \in \rho(b)$, there exists $V \in \mathcal{T}$ suth that $b \leq V \leq U$, because $a \leq b \leq V$, we have $V \in \rho_o(a)$, so $V \in \rho(a)$, because $S \rightarrow a$, it follows that S eventually belongs to U, from the arbitrariness of U, we obtain that $S \rightarrow b$, hence if a is the limit of S, $\cdot a = \{b;\ b \in \pi,\ b \geq a\}$ are also the limits of S.

Theorem 2.2 Let S be a molecular net in $L(\pi)$ and $R \subset L$ such that S is frequently in each element of R , and the intersection of two arbitrary elements of R still contains an element of R, then S has a subnet T which eventually belongs to any element of R.

The proof, being similarly with the general topology, is omitted.

Theorem 2.3 Let $L(\pi)$ be a TML, $a \in \pi$,a is the cluster of S iff, S has a subnet T converging to a.

Proof: Sufficiency is obvious, necessity can also be proved if we take the notice of the neighborhood family $\rho(a)$ satisfied the conditions of Theorem 2.2.

Definition 2.2 Let $L(\pi)$ be a TML, for any $a \in \pi$, $\rho(a)$ has finite base, $(L, \mathcal{T})$ is called to satisfy the first axiom of countability.

Theorem 2.4 $(L, \mathcal{T})$ satisfies the first axiom of countability, $a \in \pi$ is the cluster of molecular sequence S iff S has subsequences converging to a.

Proof: Sufficiency is obvious, we shall now prove the necessity. Let a be a cluster of $S = \{S_n \mid n = 1, 2, \ldots\}$ and $\{B_k \mid k = 1, 2, \ldots\}$ a countable open-neighborhood base of a and $B_k \geq B_{k+1}$, for each B_k, we take $S_{n_k} \leq B_k$ and $n_k > n_{k-1}$ (let $n_o = o$)

(a is the cluster of S, so S belongs to B frequently for $k=1,2,\ldots$), so $T=\{S_{n_k} \mid k=1,2,\ldots\}$ does be the subsequence of S which converges to a.

Similar with the convergences related to far- and Q-neighborhood, we have some corespondent conclusions as follow.

Theorem 2.5 A TML $L(\pi)$ is T_2-space iff, for each molecular net S in L, the union of limits of S has only one component.

Proof: From the Theorem 2.1, we know that if a is the limit of net S, $b \geq a$, then b is also the limit of S, so the union of limits of S is only the maximal molecule (Please read the paper [2]). We shall now prove the necessity at first. Let S be a molecular net, LimS, the union of limits of S, has more than one components, we may assume that a, b both are the distinct components of LimS, then $a \wedge b = o$. For $L(\pi)$ is T_2-space, there exist $U \in \rho_o(a)$, $V \in \rho_o(b)$ suth that $U \wedge V = o$, it is impossible for net S to belong to two disjoint sets U and V simultaneously. Contradiction. The necessity is proved.

Sufficiency. If $L(\pi)$ is not a T_2-space, then there exist $a, b \in \pi$ suth that $a \wedge b = o$, and for any $U \in \rho_o(a)$, $V \in \rho_o(b)$, we always have $U \wedge V \neq o$, because $\rho_o(a)$ and $\rho_o(b)$ both are the directed sets(directed by $\leq$), in $\rho_o(a) \times \rho_o(b)$, we make a convention: $(U,V) \geqslant (U\ ,V\ ) \Longleftrightarrow U \leq U_o$ and $V \leq V_o$, then $(\rho_o(a) \times \rho_o(b), \geqslant)$ formed a directed set, we define a net: $S=\{S(U,V),\ (U,V) \in \rho_o(a) \times \rho_o(b)\}$, in which $S(U,V)$ is a molecule selected arbitrarily in $U \wedge V$, then S converges to a, b simultaneously. Let m_a, m_b is maximal molecules which contain a, b respectively, that is, $a \leq m_a$, $b \leq m_b$, then $m_a \wedge m_b = o$ and m_a, m_b are two distinct components of the LimS.

3. Two-sided Separation Axioms

For classical concept of neighborhood has much limitations in fuzzy topological space, hence Pu Paoming, Liu Yingming introduced the important concept of Q-neighborhood in paper [1], and set up the theory of Moore-Smith convergence, which has great effect on the studies in fuzzy topological space. Later Wang Guojun introduced the concept of far-neighborhood in paper [2], the following papers occurred on separation axioms are most using the Q- and far-neighborhoods to discuss the separationity. But Wang Guojun pointed out in paper [7] that the chain of open set $\longrightarrow$ neighborhood $\longrightarrow$ interior point $\longrightarrow$ open set is still effective in fuzzy topological space, so the concept of the open-neighborhood still occupied the important position in fuzzy topological spaces. In this paper, after discussed the separationity related to open-neighborhood, now we will discuss the separationity related to open- and far-neighborhood.

To begin with, let us recall the concept of far-neighborhood occurred in the paper [2] .

Definition 3.1 Let $L(\pi)$ be a TML, $a \in \pi$, $P \in \mathcal{J}^c$, if $a \not\leq P$, then P is called a far-neighborhood for a, the family consisting of all the far-neighborhoods for a is denoted by $\eta(a)$($\mathcal{J}^c = \{A \mid A^c \in \mathcal{J}\}$).

We shall now discuss the separationity related to open- and far-neighborhood.

Definition 3.2 A TML $L(\pi)$ is called a T_{-1}'-space iff, for any a, $b \in \pi$, $a < b$, there exist $U \in \int_0(a)$, $P \in \eta(b)$ suth that $b \not\leq U$, $a \leq P$.

Wang Guojun discussed the separation axioms related to far-neighborhood in paper [4], it is easy to prove that here's

T_{-1}'-space can infer there's T_{-1}-space, and when $L(\pi)$ is a topological orthodox molecular lattice, both is equivalent, so the results on T_{-1}-space in paper [4] can be used in this T_{-1}'-space under certain conditions.

Theorem 3.1 A TML $L(\pi)$ is T_{-1}'-space iff, a is the component of $\bar{a}$ and $O(a)$ ($\bar{a}$ is the closure of a).

Proof: From definition, Theorem 3.2 in paper [4], and Theorem 1.2 in this paper, the theorem can be proved easily.

Theorem 3.2 Let $L(\pi)$ is a topological orthodox molecular lattice, $L(\pi)$ is T_{-1}'-space iff, for each molecule $a \in \pi$, a is the component of some open element.

Theorem 3.3 Let $L(\pi)$ is a topological dence molecular lattice, $L(\pi)$ is T_{-1}'-space iff every molecule belongs to the union of far-neighborhood for a, that is, for any $a \in \pi$, $a \leq \vee\{P \mid P \in \eta(a)\}$.

Definition 3.3 Let $L(\pi)$ be a TML, a, $b \in \pi$, $a \wedge b = o$, if there exist $P \in \eta(a)$, $U \in \rho_o(a)$ suth that $b \leq P$, $b \not\leq U$, or there exist $P \in \eta(b)$, $U \in \rho_o(b)$ suth that $a \leq P$, $a \not\leq U$, and $L(\pi)$ is T_{-1}'-space, then $L(\pi)$ is called a T_o'-space.

Theorem 3.4 A TML $L(\pi)$ is a T_o'-space iff, for any a, $b \in \pi$, if $a < b$, then $b \not\leq O(a)$ and $b \not\leq \bar{a}$, if $a \wedge b = o$, then $a \not\leq \bar{b}$ and $b \not\leq O(a)$ or $b \not\leq \bar{a}$ and $a \not\leq O(b)$.

The proof, being straightforward, is omitted.

Definition 3.4 A TML $L(\pi)$ is called a T_1'-space iff, for any a, $b \in \pi$, $b \not\leq a$, there exist $U \in \rho_o(a)$, $P \in \eta(b)$ suth that $b \not\leq U$, $a \leq P$..

Theorem 3.5 A TML $L(\pi)$ is a T_1'-space iff, for any $a \in \pi$, a is closed element and $O(a) = a$.

From the Theorem 3.8 in paper (4] and the Theorem 1.4, this theorem can be easily proved.

Theorem 3.6 A TML $L(\pi)$ is T_1'-space iff, all finite sets are closed sets and when $a \in \pi$, $O(a) = a$.

Proof: Considering that every molecule is finite set and the union of finite closed sets is still closed set, then using the Theorem 3.5 ,the theorem can be proved easily.

Definition 3.5 A TML $L(\pi)$ is called a T_2'-space iff, for any $a, b \in \pi$, $a \not\leq b$, there exist $P \in \eta(a)$, $U \in \rho_0(b)$ suth that $P \geq U$,

Definition 3.6 Let $L(\pi)$ is a TML and S a molecular net in $L(\pi)$, $a \in \pi$, if for any $U \in \rho_0(a)$, $P \in \eta(a)$, S belongs to U eventually and does not belong to P eventually, then we call S converging to a relative to open- and far-neighborhood, and is denoted by $S \rightarrow a$.

Theorem 3.7 A TML $L(\pi)$ is a T_2'-space iff, for every converging net, whose limit is unique.

Proof: Necessity. Let $L(\pi)$ be a T_2'-space, if there exists molecular net S suth that $a \not\leq b$ and a, b both are the limits of net S. If $a < b$, because $L(\pi)$ is T_2'-space, there exist $P \in \eta(b)$, $U \in \rho_0(a)$ suth that $P \geq U$, for $S \rightarrow a$, so S belongs to U eventually, that is, belongs to P eventually. On the other hand, $S \rightarrow b$, so for $P \in \eta(b)$, S does not belong to P eventually, we get a contradiction. If $a \wedge b = o$, we can prove it similarly.

Sufficiency. If $L(\pi)$ is not a T_2'-space, that means there exist $a, b \in \pi$, $a \not\leq b$, and for any $P \in \eta(a)$, $U \in \rho_0(b)$, we have $P \not\geq U$. Now we assume that $b < a$ at first, then obviously, we have $\eta(b) \subset \eta(a)$, $\rho_0(a) \subset \rho_0(b)$, and $(\rho_0(b), \leq)$, $(\eta(a), \geq)$ both are directed sets, for $\eta(a) \times \rho_0(b)$, we define that $(P,U) \geq (Q,V)$ iff $P \geq Q$ and $U \leq V$, then we formed a new directed set $(\eta(a) \times \rho_0(b), \geq)$ and for each (P,U), we have $P \not\geq U$, so there exists a molecule

S(P,U), suth that $S(P,U)\leq U$ and $S(P,U)\not\leq P$. Let $S=\{S(P,U), (P,U)\in \eta(a)\times \rho_o(b)\}$, then it is clear that S converges to a, b. Proving as follow: for any $U\in \rho_o(a)$ and $P\in \eta(a)$, because $\rho_o(a)\subset \rho_o(b)$, so $U\in \rho_o(b)$, for $(P,U)\in \eta(a)\times \rho_o(b)$, when $(Q,V)\geq (P,U)$, then we always have $S(Q,V)\leq V\leq U$, $S(Q,V)\not\leq Q$, so $S(Q,V)\not\leq P$, that is $S\rightarrow a$, similarly, we can prove that $S\rightarrow b$, that means there exists a molecular net S, which has more than one limit at least. If $a\wedge b = o$, similar with the Theorem 2.5, we can also prove it.

Definition 3.7 A TML $L(\pi)$ is called a T_3'-space iff, for any molecule a, closed set A, $a\not\leq A$, if there exist $P\in \eta(a)$, $U\in \rho_o(A)$ suth that $P\geq U$.

It is clear that $T_3' + T_1' \Rightarrow T_2'$.

Theorem 3.8 Let $(L|E, \mathcal{T}\wedge E)$ is the topological sublattice of $(L(\pi),\mathcal{T})$, if $(L(\pi),\mathcal{T})$ is T_i (i= -1,o,1,2,3)-space, then $(L|E, \mathcal{T}\wedge E)$ also is T_i(i= -1,o,1,2,3)-space.

The proof is omitted.

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