# FUZZY INNER PRODUCT SPACE $V_n$

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ABSTRACT

In this paper a fuzzy inner product space and a fuzzy linear normed space are definited and their properties are discussed, and we prove: there exist and there exist only a standard orthogonal basis in each fuzzy inner product space $V_n$.

Keywords: Fuzzv inner product space of $V_n$. Fuzzy liner nomed space of $V_n$. Orthogonal basis of $V_n$. Standard orthogonal basis of $V_n$. Simple standard orthogonal basis of $V_n$.

## I. FUZZY INNER PRODUCT SPACE $V_n$

For definition of term and sign used in this paper see [1] and [5].

Definition 1.1 Let V be a fuzzy semilinear space. If, for an arbitrary pair of elements u and v, there is a number (u,v) of [0,1] such that setisfies:

1) (u,v) = (v,u)

2) $(ku,v)=k(u,v)$ $k\in[0,1]$

3) $(u+v,w)=(u,w)+(v,w)$, $w\in V_n$

4) $(u,u)=0$ iff $u=\theta$

then $V_n$ is called a fuzzy inner product space, $(u,v)$ is called the fuzzy inner product of u and v.

<u>Proposition 1.1</u> In fuzzy inner product space V the following formulas hold:

1) $(ku, hv)=kh(u,v)$, $k,h\in[0,1]$

2) $(u,v+w)=(u,v)+(u,w)$

3) $(u,kv+hw)=k(u,v)+h(u,w)$, $k,h\in[0,1]$

4) If u or v is $\theta$, then $(u,v)=0$

5) $(\sum_{i=1}^{m} k_i u_i, \sum_{j=1}^{n} h_j v_j)=\sum_{i=1}^{m}\sum_{j=1}^{n} k_i h_j (u_i,v_j)$, $k_i,h_j\in[0,1]$

In the finite spanning innen product space V, Let $\{e_1, \dots, e_n\}$ be a basis of fuzzy inner product space V, for arbitrary $u\in V$, $v\in V$ if $u=x_1e_1+\dots+x_ne_n$, $v=y_1e_1+\dots+y_ne_n$ then $(u,v)=(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j)=\sum_{i=1}^{n}\sum_{j=1}^{n}(e_i,e_j)x_i y_j$. Let $a_{ij}=(e_i,e_j)$ and $A=(a_{ij})_{m\times n}$ then

$$X\ A\ Y^T=(x_1 \ \dots \ x_n)\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i y_j=(u,v)$$

that is $(u,v)=XAY^T$, where $X=(x_1, \dots, x_n)$ and $Y=(y_1, \dots, y_n)$.

<u>Definition 1.2</u> Let $\{e_1,\dots,e_n\}$ be a basis of a fuzzy inner product space V and $a_{ij}=(e_i,e_j)$ then $A=(a_{ij})$ is called a metric matrix of V under the basis $\{e_1,\dots,e_n\}$.

Proposition 1.2 The metric matrix A of fuzzy inner product space V under some basis is symmetrie.

Proposition 1.3 The inner product of arbitrary vectors u and v of fuzzy inner product space V are denoted by the coordinates of fuzzy vector and the metric matrix.

Theorem 1.1 Let two bases $\{e_1,\dots,e_n\}$ and $\{v_1,\dots,v_n\}$ of fuzzy inner product space V and A is a metric matrix of V under the basis $\{e_1,\dots,e_n\}$ and B is a metric matrix of V under the basis $\{v_1,\dots,v_n\}$ if C is a transition matrix from $\{e_1,\dots,e_n\}$ to $\{v_1,\dots,v_n\}$ that is $(v_1,\dots,v_n)=(e_1,\dots,e_n)C$ then $B=(b_{ij})_{n\times n}=((v_i,v_j))_{n\times n}=C^TAC$.

Definition 1.3 For two fuzzy matrices A and B if there is a fuzzy matrix C such that $B=C^TAC$ then B and A is called similar.

Proposition 1.4 The similar relation of fuzzy matrices possess:

1) reflexivity: A and A are similar.

2) transitivity: if A and B are similar, B and C are similar then A and C are similar.

Theorem 1.2 The metric matrices of fuzzy inner product space V under different bases are similar.

## II. THE FUZZY INNER PRODUCT SPACE $V_n$

Proposition 2.1 In fuzzy semilinnear space $V_n$ if for any fuzzy vectors $u=(a_1,\dots,a_n)$ and $v=(b_1,\dots,b_n)$, we define $(u,v)=uv^T=(a_1,\dots,a_n)(b_1,\dots,b_n)^T=\bigvee_{i=1}^{n}(a_i\wedge b_i)$ as the inner product of u and v, then $V_n$ is a inner product space.

Proposition 2.2 Under the operation of inner product $(u,v)=u^Tv$, the fuzzy semilinear space $V^n$ forms also a fuzzy inner product space.

In this paper the following discusses are only confined to $V_n$, the descusses of $V^n$ are all similar.

Definition 2.1 Let V be a fuzzy semilinear space, if for every element u of V, there is a number $\|u\|$ corresponding to it that satisfies the following condition:

1) $1 \geq \|u\| \geq 0$, $\|u\|=0$ iff $u=\theta$

2) $\|ku\|=k\|u\|$, $k\in[0,1]$

3) $\|u+v\| \leq \|u\|+\|v\|$

then V is called fuzzy linear normed space, and $\|u\|$ is called the norm of u.

Proposition 2.3 In $V_n$ let $\|u\|=(u,u)$ then $V_n$ is a fuzzy linear normed space.

Proposition 2.4 For an arbitrary $u=(a_1,\ldots,a_n)\in V_n$ then $\|u\|=\max\{a_1,\ldots,a_n\}$.

Theorem 2.1 For arbitrary $u,v\in V_n$, Cauchy-Буняковский inequality stands: $(u,v)\leq\|u\|\cdot\|v\|$.

Proposition 2.5 In a fuzzy linear normed space $V_n$ the following hold:

1) $\|u+v\|\leq\|u\|+\|v\|$

2) $\|u\|^k=\|u\|$, $k\in N$

3) $\|u+v\|^2\leq\|u\|^2+\|v\|^2$

4) $\|u+v+\ldots+w\|^r\leq\|u\|^r+\|v\|^r+\ldots+\|w\|^r$, $r\in N$

## III. A STANDARD ORTHOGONAL BASIS OF $V_n$

Definition 3.1 Let $u,v\in V_n$, if $(u,v)=0$, then u and v are called orthogonal.

A vectors group of consisting of non—zero vectors is called an orthogonal group if every two vectors of it are orthogonal.

Proposition 3.1 In $V_n$ there stand:

1) $\|u+v\|=\|u\|+\|v\|$ iff $(u,v)=0$

2) $\|u+v\|^2=\|u\|^2+\|v\|^2$ iff $(u,v)=0$

3) $\|u+v+\ldots+w\|^r=\|u\|^r+\|v\|^r+\ldots+\|w\|^r$ , $r\in N$

iff u,v,...,w are orthogonal each other.

Proposition 3.2 Let u be a element of $V_n$ then $S=\{v\,|\,(u,v)=0\ ,\ v\in V_n\}$ is said to be the maxium orthogonal subspace of u.

Definition 3.2 Let $W_1$ and $W_2$ be two orthogonal subspaces of $V_n$. If for arbitrary $u\in W_1$ and $v\in W_2$ there is $(u,v)=0$ then the subspaces $W_1$ and $W_2$ is called orthogonal.

Proposition 3.3 Let S be a subspace of $V_n$, then the set of all vectors to each of which S is orthogonal is subspace, which is called the orthogonal subspace of S.

Definition 3.3 In $V_n$ a vector is called identity norm vector if its norm is 1. If vectors of a identity norm vector group of $V_n$ are orthogonal mutually, then it is called a identity normed orthogonal group. For the sake of convenience, a identity normed vector is also called orthogonal.

The definition of a maximal independent vector group of $V_n$ see definition 1.4 of (4).

Proposition 3.4 1) A non-zero orthogonal spanning vector group of $V_n$ is a maximal independent group of $V_n$.

2) An identity normed orthogonal spanning vector group of $V_n$

is a maximal independent group of $V_n$.

3) The numbers of vectors of orthogonal spanning vector group of $V_n$ are equal. number of vectors of identity normed orthogonal spanning vector group of $V_n$ are equal.

Proposition 3.5 Let a set $\{u_1,\ldots,u_n\}$ which $u_i \in V_n$ $(i=1,\ldots,t)$ be an orthogonal vector group of $V_n$ then $S=L(u_1,\ldots,u_n)$ is a subspace of $V_n$ and is called a orthogonal subspace of $V_n$. $\{u_1,\ldots,u_n\}$ is called a orthogonal basis of $S$.

If $\{u_1,\ldots,u_n\}$ is a identity normed orthogonal vector group of $V_n$ then $S=L(u_1,\ldots,u_n)$ is called a standard orthogonal subspace of $V_n$ and $\{u_1,\ldots,u_n\}$ is called a standard orthogonal basis of $S$.

Proposition 3.6 If $\{u_1,\ldots,u_n\}$ is a standard orthogonal basis of $L(u_1,\ldots,u_n)$ then $\{u_2,\ldots,u_n\}$ is a standard orthogonal basis of $L(u_2,\ldots,u_n)$.

Proposition 3.7 Let $u_1=(0,a_{12},\ldots,a_{1n}),\ldots,u_t=(0,a_{t2},\ldots,a_{tn})$ is a standard orthogonal basis of $L(u_1,\ldots,u_t)$ if and only if $u_1^*=(a_{12},\ldots,a_{1n}),\ldots,u_t^*=(a_{t2},\ldots,a_{tn})$ is a standard orthogonal basis of $L(u_1^*,\ldots,u_t^*)$.

Definition 3.4 For two bases $\{u_1,\ldots,u_n\}$ and $\{v_1,\ldots,v_n\}$ of $W$ if $L(u_1,\ldots,u_n)=L(v_1,\ldots,v_n)$ then two bases are called identical.

Theorem 3.1 There exists exactly one standard orthogonal basis in each fuzzy linear normed space $V_n$.

Theorem 3.2 The subspace producted by some vector of the standard orthogonal basis of $V_n$ is a standard orthogonal subspace of $V_n$.

## IV. A SIMPLE VECTOR AND A COMPOUND VECTOR

Definition 4.1 Let $W$ be a finite spanning subspace of $V_n$. For $u \in W$ if there is non-ordered relation "$\leq$" $v, w \in W$ such that $u=v+w$ then $u$ is called a compound vector of $W$ otherwise $u$ is called a simple vector of $W$.

Proposition 4.1 Let $W$ be a finite spanning subspace of $V_n$.

1) If $u \in W$ is a compound vector of $W$ then $u$ is a compound vector of $V_n$.

2) If $u \in W$ is a simple vector of $V_n$ then $u$ is a simple vector of $W$.

Theorem 4.1 (the judgment theorem of a compound vector) Let $u \in W$.

1) $u$ is a compound vector of $W$ if and only if there are $v \in W$ and $w \in W$ which are non-ordered relation "$\leq$" such that

$$u=v+w \qquad (1)$$

2) Let (1) hold and $u=(a_1,\dots,a_n)$, $v=(b_1,\dots,b_n)$ and $w=(c_1,\dots,c_n)$ $v$ and $w$ are non-ordered relation "$\leq$" if and only if at least there is a coordinate $b_{i_0}$ such that $0 \leq b_{i_0} < a_{i_0}$, $i_0 \in \{1,\dots,n\}$ and at least there is a coordinate $c_{j_0}$ such that $0 \leq a_{j_0} < c_{j_0}$, $j_0 \in \{1,\dots,n\}$ and $i_0 = j_0$.

Proposition 4.2 Let $W$ be a finite spanning subspace of $V_n$. $u \in W$ is a simple vector of $W$ if and only if for arbitrary $v, w \in W$ if $u=v+w$ then $v$ and $w$ are ordered relation "$\leq$".

Notice: the simple vector and compound vector relate to subspace $W$. Specially we have:

Proposition 4.3 $u$ is a simple vector of $V_n$ if and only if u is Like the vector

$$u=(0,\dots,0,a\ ,0,\dots,0),\ a\in[0,1]$$

## V. A SIMPLE STANDARD ORTHOGONAL BASIS OF $V_n$

Definition 5.1 If a subspace S of $V_n$ possesses a standard orthogonal basis, which every vector is a simple vector of $V_n$, then S is called a simple standard orthogonal subspace of $V_n$ and the basis is called a simple standard orthogonal basis of S.

Proposition 5.1 u is a identity normed simple vector if and ionly if u is like the vector

$$u_i=(0,\dots,0,1,0,\dots,0),\ (i=1,\dots,n) \qquad (2)$$

where $u_i$ is a vector which coordinate $i$ is 1 and other coordinated are zero.

Theorem 5.1 There exist exactly one simple standard orthogonal basis in each fuzzy linear normed space $V_n$.

The simple standard orthogonal basis and the standard orthogonal basis of $V_n$ are identical basis which formed by $\{u_1,\dots,u_n\}$ of (2).

The metrie matrix of space $V_n$ under the basis $u_1,\dots,u_n$ is a identity matrix.

Proposition 5.2 Lte $\{u_1,\dots,u_n\}$ are like the vector of (2) for arbitrary $v,w\in V_n$

1) $v=(v,u_1)u_1+\dots+(v,u_n)u_n$

2) $(v,w)=(v,u_1)(w,u_1)+\dots+(v,u_n)(w,u_n)$

Proposition 5.3 Let $S_1$ and $S_2$ are two orthogonal subspace of $V_n$. If $S_1$ and $S_2$ are orthogonal then $S_1\cap S_2=\{\theta\}$.

Proposition 5.4 Let S be a simple standard orthogonal subspace of $V_n$ then the basis of S is composed by some vectors of (2).

Proposition 5.5 The simple standard orthogonal subspaces of $V_n$ have $2^n-1$

Proposition 5.6 Let S be a simple standard orthogonal subspace then $T=V_n-S+\{\theta\}$ is also a simple standard orthogonal subspace of $V_n$ and $S\cap T=\{\theta\}$.

Definition 5.2 Let $S_1$ and $S_2$ are two simple standard orthogonal subspaces of $V_n$. If $S=S_1+S_2$ and $S_1\cap S_2=\{\theta\}$ then S is called direct sum of $S_1$ and $S_2$ and is denoted $S=S_1\dot{+}S_2$.

Theorem 5.2 Let $S_1$ and $S_2$ be two simple standard orthogonal subspaces of $V_n$ and $S=S_1+S_2$ then $S=S_1\dot{+}S_2$ iff $S_1$ and $S_2$ are orthogonal.

Theorem 5.3 Let S be a simple standard orthogonal subspace of $V_n$ and $T=V_n-S+\{\theta\}$ then $V_n=S\dot{+}T$ and T is called a direct complementary space of S and is denoted $S^{\perp}=T$, that is $S^{\perp}=V_n-S+\{\theta\}$.

Proposition 5.7 Let S and W be two simple standard orthogonal subspaces of $V_n$ then $(S+W)^{\perp}=S^{\perp}+W^{\perp}$.

Proposition 5.8 The sum of two simple standard orthogonal subspaces of $V_n$ is still a simple standard orthogonal subspace of $V_n$.

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