THE DIRECT SUM OF FUZZY RINGS

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Abstract

Many papers about Fuzzy algebra have been published since A.Rosenfeld presented the concept of Fuzzy subrings in 1971. As further work in this respect, we study the direct sum of Fuzzy subrings in this paper, introduce the concept of projective Fuzzy sets and marginal Fuzzy sets of direct sum rings, and obtain a necessary and sufficient condition that a Fuzzy subring on the direct sum ring may be represented by the direct sum of Fuzzy subrings on each ring. Finally, a similar discussion of t-norm Fuzzy subrings is suggested.

Key words: Fuzzy subring, Direct sum ring, projective fuzzy sets, Marginal Fuzzy Sets.

一 PRELIMINARIES

Let us recall some basic definitions and well known result.

Definition 1.1: Let R be a ring, μ is a fuzzy subset of R. μ is called a fuzzy subring of R if it satisfies following conditions:

(1) $\mu(x.y) \geqslant \mu(x) \wedge \mu(y)$ for any x and y in R.

(2) $\mu(x+y) \geqslant \mu(x) \wedge \mu(y)$ for any x and y in R.

(3) $\mu(x)=\mu(-x)$ for any x in R.

Corollary : If μ is a fuzzy subring of R. Then

$\mu(x) \leq \mu(0)$ for any x in R.

THEOREM 1.1: Suppose R is a ring, and μ is a fuzzysubset of R. Then following conditions are equivalence,

(1) μ is a fuzzy subring of R.

(2) for any x and y in R, $\mu(x\cdot y)\geqslant\mu(x)\wedge\mu(y)$

$$\mu(x-y)\geqslant\mu(x)\wedge\mu(y)$$

(3) for any x and y in R,

$$\min\{\mu(x\cdot y),\mu(x-y)\}\geqslant\mu(x)\wedge\mu(y).$$

Definition 1.2: Suppose that μ is a fuzzy subring of the ring R. μ is called an fuzzy left (or right) ideal if for any x and y in R, $\mu(x\ y)\geqslant\mu(y)$ (or $\mu(x)$). If μ is not only an fuzzy left ideal, but alsoan fuzzy right ideal, it is called an fuzzy bi-ideal. For the sake of simplicity μ is called an fuzzy ideal.

Notes: The above definitions and results may be found in [2]

二 THE DIRECT SUM OF FUZZY SUBRINGS

For the sake ofsimplicity, the direct sums of only two rings are discussed in this paper, but we may completely analogize on the direct sums of any finite rings. Without specified statement in this section, R_1 and R_2 are always rings.

Definition 2.1; Let X and Y be two sets. μ_1 and μ_2 are the fuzzy sets of X and Y, respectively. Define a fuzzy set μ of XxY : $\forall(x,y)\in X\times Y$, $\mu(x,y)=\mu_1(x)\wedge\mu_2(y)$. μ is called the direct product of μ_1 and μ_2, denote $\mu=\mu_1\otimes\mu_2$.

Theorem 2.1: Suppose R_1 and R_2 are two rings, μ_1 and μ_2 are fuzzy subrings of R_1 and R_2, respectively. Then the direct sum μ of μ_1 and μ_2 is a fuzzysubring of the ring $R_1\oplus R_2$, μ is called direct sum fuzzy subring. denote $\mu = \mu_1 \oplus \mu_2$ for the sake of consistency.

Proof: For any x and y in $R_1 \oplus R_2$, take $x=(x_1, x_2)$ $y=(y_1, y_2)$

Then $\mu(x \cdot y) = \mu(x_1 \cdot y_1, x_2 \cdot y_2) = \mu_1(x_1 \cdot y_1) \wedge \mu_2(x_2 \cdot y_2)$

$$\geqslant \mu_1(x_1) \wedge \mu_1(y_1) \wedge \mu_2(x_2) \wedge \mu_2(y_2)$$
$$= \mu_1(x_1) \wedge \mu_2(x_2) \wedge \mu_1(y_1) \wedge \mu_2(y_2)$$
$$= \mu(x_1, x_2) \wedge \mu(y_1, y_2) = \mu(x) \wedge \mu(y)$$

$$\mu(x-y) = \mu(x_1 - y_1, x_2 - y_2) = \mu_1(x_1 - y_1) \wedge \mu_2(x_2 - y_2)$$
$$\geqslant \mu_1(x_1) \wedge \mu_1(y_1) \wedge \mu_2(x_2) \wedge \mu_2(y_2)$$
$$\mu(x) \wedge \mu(y)$$

By Theorem 1.1, we see that μ is a fuzzy subring of $R_1 \oplus R_2$.

Q.E.D.

Theorem 2.2. If μ_1 and μ_2 are fuzzy (or left, or right) ideals of the rings R_1 and R_2, respectively. Then $\mu = \mu_1 \oplus \mu_2$ is also a fuzzy (or left, or right) ideal of ring $R_1 \oplus R_2$.

Proof : By Theorem 2.1 , we know that μ is a fuzzy subring of $R_1 \oplus R_2$. If μ_1 and μ_2 are fuzzy left ideals of R_1 and R_2, respectively. Then we only need to prove $\mu(x \cdot y) \geqslant \mu(y)$ holds for any x and y in $R_1 \oplus R_2$.

In fact, take $x=(x_1, x_2)$, $y=(y_1, y_2)$

$$\mu(x \cdot y) = \mu(x_1 \cdot y_1, x_2 \cdot y_2) = \mu_1(x_1 \cdot y_1) \wedge \mu_2(x_2 \, y_2)$$
$$\mu_1(y_1) \wedge \mu_2(y_2) = \mu(y)$$

Therefore μ is an fuzzy left ideal of $R_1 \oplus R_2$.

Similar, we can prove the situations of fuzzy right ideal and fuzzy ideal .

Q.E.D.

Conversely, reads will naturely ask: if μ is a fuzzy subring or ideal of $R_1 \oplus R_2$, are there subrings or ideals of R_1 and R_2 respectively, such that $\mu = \mu_1 \oplus \mu_2$? In generally speaking, the answer is no (see following counterexample). If we want above statement holds, what conditions will be asked? Now let us study this problem.

Counterexample: Suppose that $Z_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ is the ring of residues module 3 . Consider its fuzzy subset μ :

$$\mu(x,y) = \begin{cases} 1 & (x,y) = (\bar{0},\bar{0}) \\ 0.8 & (x,y) \in \{(\bar{1},\bar{1}),\ (\bar{2},\bar{2})\} \\ 0.5 & \text{others} \end{cases}$$

It is easily to check that μ is a fuzzy subring of $Z_3 \oplus Z_3$, But it can not be represented the direct sum of two fuzzy subrings of Z_3 .

Suppose μ is a fuzzy subset of $R_1 \oplus R_2$,let $\mu_1(x) = \bigvee_{y \in R_2} \mu(x,y)$ $\mu_2(y) = \bigvee_{x \in R_1} \mu(x,y)$ for any x in R_1 and y in R_2. Then μ_1 and μ_2 are fuzzy subsets of R_1 and R_2 , and called projections of μ on R_1 and R_2 respectively .

Moreover, for each x in R_1 and y in R_2 , take $\mu_1(x) = \mu(x,0)$ $\mu_2(y) = \mu(0,y)$. μ_1 and μ_2 are also fuzzy subsets, as analogues of the marginal distribution in probability Theory, μ_1 and μ_2 are called marginal fuzzy subsets of μ .

Lemma : Suppose X is a set, f is a function on X, and α is a a given real constant number. Then

$$\alpha \wedge (\sup_{x \in X} f(x)) = \sup_{x \in X} (\alpha \wedge f(x))$$

Proof :Since this lemma is the distributive law about Zadeh'S operator.the ptoof is omitted.

Theorem 2.3 : Suppose that R_1 and R_2 are rings ,μ is a fuzzy subring of $R_1 \oplus R_2$ .Then projective fuzzy sets μ_1 and μ_2 are fuzzy subrings of R_1 and R_2, respectively.

Proof : We only prove that μ_1 is a fuzzy subring of R_1,similar proof about μ_2 is omitted.

for any x_1 and x_2 in R_1, $\mu_1(x_1 \cdot x_2) = \bigvee_{y \in R_2} \mu(x_1 \cdot x_2, y)$

$$\geqslant \bigvee_{y_1, y_2 \in R_2} \mu(x_1 \cdot x_2, y_1 \cdot y_2) = \bigvee_{y_1, y_2 \in R_2} \mu((x_1, y_1) \cdot (x_2, y_2))$$

$$\geqslant \bigvee_{y_1, y_2 \in R_2} [\mu(x_1, y_1) \wedge \mu(x_2, y_2)]$$

$$= [\bigvee_{y_1 \in R_2} \mu(x_1, y_1)] \wedge [\bigvee_{y_2 \in R_2} \mu(x_2, y_2)] \quad \text{(By Lemma)}$$

$$= \mu_1(x_1) \wedge \mu_1(x_2)$$

Hence $\mu_1(x_1 \cdot x_2) \geqslant \mu_1(x_1) \wedge \mu_1(x_2)$

Similarly, we can get $\mu_1(x_1 - x_2) \geqslant \mu_1(x_1) \wedge \mu_1(x_2)$

By Theorem 1.1, we see that μ_1 is a fuzzy subring of R_1. Q.E.D.

Theorem 2.4 :Suppose R_1 and $R_2$are rings ,μ is an fuzzy (or left, or right) ideal, then projective fuzzy sets μ_1 and μ_2 are fuzzy (or left,or right)ideals of R_1 and R_2 respectively.

Proof :By Theorem 2.3, we know that μ_1 and $\mu_2$are fuzzy subrings of R_1 and R_2 respectively. If μ is an fuzzy ideal of R_1 R_2, we only need to prove that : for any x_1 and x_2 in R_i , $\mu_i(x_1 \cdot x_2) \geqslant \max\{\mu_i(x_1), \mu_i(x_2)\}$ $i=1,2$.

In fact $\forall$ x_1, $x_2 \in R_1$

$$\mu_1(x_1 \cdot x_2) = \bigvee_{y \in R_2} \mu(x_1 \cdot x_2, y) \geqslant \bigvee_{y_1, y_2 \in R_2} \mu(x_1 \cdot x_2, y_1 \cdot y_2)$$

$$= \bigvee_{y_1, y_2 \in R_2} \mu((x_1, y_1) \cdot (x_2, y_2))$$

$$\geqslant \bigvee_{y_1,y_2\in R_2}[\mu(x_1,y_1)\wedge \mu(x_2,y_2)]$$
$$=[\bigvee_{y_1\in R_2}\mu(x_1,y_1)]\wedge[\bigvee_{y_2\in R_2}\mu(x_2,y_2)] = \mu_1(x_1)\wedge\mu_1(x_2)$$

Therefore μ_1 is an fuzzy ideal of R_1.

Similarly, other situations can be proved. Q.E.D.

Theorem 2.5 : If μ is a fuzzy subring (or ideal) of $R_1 \oplus R_2$. Then marginal fuzzy subsets μ_1' and μ_2' of μ are fuzzy subrings (or ideals) of R_1 and R_2 respectively.

Proof : Since μ is a fuzzy subring of $R_1 \oplus R_2$, for any x_1 and x_2 in R_1,
$$\mu_1'(x_1\cdot x_2) = \mu(x_1\cdot x_2, 0) = \mu(x_1\cdot x_2, 0\cdot 0)$$
$$\geqslant \mu(x_1,0)\wedge\mu(x_2,0) = \mu_1'(x_1)\wedge\mu_1'(x_2)$$
$$\mu_1'(x_1-x_2) = \mu(x_1-x_2,0) = \mu(x_1-x_2,0-0)$$
$$\geqslant \mu(x_1,0)\wedge\mu(x_2,0) = \mu_1'(x_1)\wedge\mu_1'(x_2)$$

As we know, μ_1' is a fuzzy subring of R_1.

We can prove μ_2' is a fuzzy subring of R_2 in similar way.

If μ is an fuzzy ideal of $R_1 \oplus R_2$, we only need to prove: for any x_1 and x_2 in R_i, $\mu_i'(x_1\cdot x_2) \geqslant \mu_i'(x_1)\vee\mu_i'(x_2)$ $i=1,2$.

In fact, $\mu_1'(x_1\cdot x_2) = \mu(x_1\cdot x_2,0) = \mu(x_1\cdot x_2, 0.0)$
$$\geqslant \mu(x_1,0)\vee\mu(x_2,0) = \mu_1'(x_1)\vee\mu_1'(x_2)$$

Hence μ_1' is an fuzzy ideal of R_1.

Similarly, we may prove μ_2' is an fuzzy ideal of R_2. Q.E.D.

Lemma : Suppose R_1 and R_2 are rings, μ is a fuzzy subring (or ideal) of $R_1 \oplus R_2$, μ_1', μ_2' and μ_1, μ_2 are marginal fuzzy subrings and projective fuzzy subrings of μ, respectively.

Then $\mu_1' \oplus \mu_2' \subseteq \mu \subseteq \mu_1 \oplus \mu_2$

Proof : For any x in R_1 and y in R_2,
$$\mu_1'(x) = \mu(x,0) \qquad \mu_2'(y) = \mu(0,y)$$

So $\mu(x,y)=\mu[(x,0)+(0,y)]\geqslant\mu(x,0)\wedge\mu(0,y)$

$$=\mu_1'(x)\wedge\mu_2'(y) = (\mu_1'\oplus\mu_2')(x,y)$$

Therefore $\mu_1'\oplus\mu_2'\subseteq\mu$

Since $\mu_1(x)=\bigvee_{y\in R_2}\mu(x,y)\geqslant\mu(x,y)$,

$$\mu_2(y)=\bigvee_{x\in R_1}\mu(x,y)\geqslant\mu(x,y)$$

$$\mu(x,y)\leq\mu_1(x)\wedge\mu_2(y)$$

Hence $\mu\subseteq\mu_1\oplus\mu_2$,

As we see that $\mu_1'\oplus\mu_2'\subseteq\mu\subseteq\mu_1\oplus\mu_2$. Q.E.D.

Theorem 2.6 : Suppose μ is a fuzzy subring (or ideal), Then μ may be represented the direct sum of two fuzzy subrings (or ideals) of R_1 and R_2 if and only if $\mu_1'\oplus\mu_2'=\mu_1\oplus\mu_2$, where μ_1', μ_2', μ_1 and μ_2 are defined as above.

Proof : By Lemma, we know that sufficient condition holds.

Necessity : Suppose $\mu=\mu_1''\oplus\mu_2''$, where μ_1'' and μ_2'' are fuzzy subrings (Or ideals) of R_1 and R_2 respectively. Then $\mu_i''(x)\leq\mu_i''(0)$ for any x in R_i, hence $\bigvee_{x\in R_i}\mu_i''(x)=\mu_i''(0)$ i=1,2.

So $\mu_1'(x)=\mu(x,0)=(\mu_1''\oplus\mu_2'')(x,0)=\mu_1''(x)\wedge\mu_2''(0)$

$$=\mu_1''(x)\wedge[\bigvee_{y\in R_2}\mu_2''(y)]=\bigvee_{y\in R_2}[\mu_1''(x)\wedge\mu_2''(y)]$$

$$=\bigvee_{y\in R_2}\mu(x,y)=\mu_1(x)$$

Therefore $\mu_1'(x)=\mu_1(x)$ for any x in R_1, i.e. $\mu_1'=\mu_1$

Similarly, we can obtain $\mu_2'=\mu_2$

$\mu_1'\oplus\mu_2'=\mu_1\oplus\mu_2$ holds. Q.E.D.

三 THE DIRECT SUM OF t-NORM FUZZY SUBRINGS

We have studied the structures of direct sum of fuzzy subrings about Zadeh's operator ($\vee$, $\wedge$) in section(二).

In this section we will discuss that structures about t-norm operators. Our notions and terminologies whish we have not given can be found in [1] and [3]. All proofs are omitted, since the method of the proof is similar to that of those Theorems in last section.

Definition 3.1 : Suppose R is a ring, μ is a fuzzy subset of R. If μ satisfies following conditions, μ is called a fuzzy subring about t-norm T.

(1) $\mu(x\cdot y) \geqslant T(\mu(x),\mu(y))$ for any x and y in R,

(2) $\mu(x+y) \geqslant T(\mu(x),\mu(y))$ for any x and y in R,

(3) $\mu(-x) = \mu(x)$ for any x in R,

(4) $\mu(0) = 1$ 0 is the zero element in R.

Definition 3.2 : Suppose μ_1 and μ_2 are a fuzzy subset of R_1 and R_2 respectively. Define a fuzzy subset μ of $R_1 \oplus R_2$ by $\mu(x,y) = T(\mu_1(x),\mu_2(y))$ for any (x,y) in $R_1 \oplus R_2$. μ is called the direct sum of μ_1 and μ_2 about t-norm T, denote $\mu = \mu_1 \oplus_T \mu_2$.

Theorem 3.1: Let μ_1 and μ_2 be fuzzy subrings about t-norm T of R_1 and R_2 respectively. Then $\mu = \mu_1 \oplus_T \mu_2$ is a fuzzy subring of $R_1 \oplus R_2$ about t-norm T, too.

Theorem 3.2 : Let μ_1 and μ_2 be fuzzy left (or right) ideals about t-norm T of R_1 and R_2 respectively. Then $\mu = \mu_1 \oplus_T \mu_2$ is an fuzzy left (or right) ideal about t-norm T of $R_1 \oplus R_2$.

Lemma : Suppose T is a continuously t-norm, for any subset X of $I=[0,1]$, and any α in I, then

$$\sup_{x\in X} T(\alpha, x) = T(\alpha, \sup_{x\in X} x)$$

Theorem 3.3 : Suppose R_1 and R_2 are rings, μ is a fuzzy subrings about t-norm T of $R_1 \oplus R_2$, and T is continuously t-norm. Then projective fuzzy sets μ_1 and μ_2 of μ are fuzzy subrings about t-norm T of R_1 and R_2 respectively.

Theorem 3.4 : If μ is fuzzy subring about t-norm T of $R_1 \oplus R_2$. And T is continuously t-norm, μ_1' and μ_2' are marginal fuzzy subsets of μ on R_1 and R_2 respectively. Then μ_1' and μ_2' are fuzzy subrings about t-norm T of R_1 and R_2 respectively.

Theorem 3.5 :Suppose T is a continuously t-norm ,and for any α in $[0,1]$,$T(\alpha,\alpha) \geqslant \alpha$,μ is a fuzzysubring about t-norm T of $R_1 \oplus R_2$. Then μ may be represented by the direct sum about t-norm T of fuzzy subrings of R_1 and R_2 if and only if if $\mu_1' \oplus_T \mu_2' = \mu_1 \oplus_T \mu_2$ .Where μ_1' and μ_2' are marginal fuzzy subrings of μ, and μ_1 and μ_2 are projective fuzzy subrings.

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