FUZZY UNIVERSAL ALGEBRA

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ABSTRACT

In this note we define the concept of fuzzy universal algebras and outline their simple properties with respect to homomorphisms, fuzzy equivalance relation and fuzzy congruence relations. It is shown that given a fuzzy congruence relation on an algebra, a subclass of fuzzy subsets of the carrier space of the algebra can be obtained and that this subclass of fuzzy subsets with the induced fuzzy operations is an algebra similar to and is isomorphic with the given algebra.

1. INTRODUCTION

In the past fifteen years, groups [9], vector spaces [6], rings [12] etc. have been fuzzified and studied in some details. Also in [8], E.G.Manes proposed a class of fuzzy theories where he mentioned the fuzzification of universal algebra. Since groups, rings, vector spaces etc. are particular classes of universal algebra, it seems appropriate to take a closer look at the fuzzification of universal algebras and to study them in more depth. In this note we make such an attempt, show the unification of [9], [6] and [12] and also pave the way for further studies in fuzzification of other algebraic structures like lattices, semigroups, modules etc. We shall not give the full technical details here. They will be published elsewhere. We only outline the results here. We refer the reader to the standard references on Universal Algebra by P.M. Cohn [2] and G. Gratzer [5], and on lattice theory by G. Birkoff [1] and G. Szasz [11].

Let I be the unit interval. For any non-empty set X, let $\mathfrak{F}(X)$ denote the set of all fuzzy subsets of X. Let A be a finitary algebra denoted by [S,F] where S is the underlying set called the carrier space of A and F is the set of algebraic operations on S. Every operation $f_\alpha \in F$ on S induces an operation on $\mathfrak{F}(S)$ as follows:

Let $f_\alpha : S^{n(\alpha)} = S X S X \ldots X S \rightarrow S$

$(x_1, x_2, \ldots, x_{n(\alpha)}) \rightarrow x$

$\mathfrak{f}_\alpha : (\mathfrak{F}(S))^{n(\alpha)} : \mathfrak{F}(S) X \mathfrak{F}(S) X \ldots X \mathfrak{F}(S) \rightarrow \mathfrak{F}(S)$

$(\mu_1, \mu_2, \ldots, \mu_{n(\alpha)}) \rightarrow \mu$

where for any $x \in S$,

$$\mu(x) = \sup_{x = f_\alpha(x_1, x_2, \ldots, x_{n(\alpha)})} \{\mu_1(x_1) \wedge \mu_2(x_2) \ldots \wedge \mu_{\mu n(\alpha)}(x_{n(\alpha)})\}$$

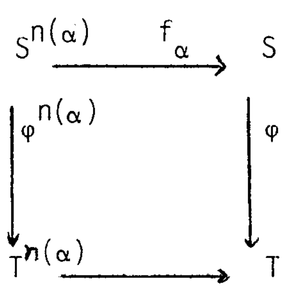
See the fuzzification principle 6.1 [8] or Zadeh's extension principle [14]. Let

$\mathcal{F}$ denote $\{\mathfrak{f}_\alpha : f_\alpha \in F\}$

Our main results are the following.

1. If $\mathfrak{F}\mathcal{P}(S)$ denotes the subset of fuzzy points (see [10]) in $\mathfrak{F}(S)$ then $\mathcal{A} = [\mathfrak{F}\mathcal{P}(S), \mathcal{F}]$ is an algebra similar to and isomorphic to the given algebra A = [S,F].

2. If A = [S,F] and B = [T,F] are similar algebras and φ is a mapping from A to B and if the induced mapping from $\mathfrak{F}(S)$ to $\mathfrak{F}(T)$ (see [13]) is also denoted by φ, then the following diagram



commutes for every $f_\alpha \in F$ if and only if the diagram

$$\begin{array}{ccc}
(\mathfrak{F}(S))^{n(\alpha)} & \xrightarrow{\mathfrak{f}_\alpha} & \mathfrak{F}(S) \\
\Big\downarrow \varphi^{n(\alpha)} & & \Big\downarrow \varphi \\
(\mathfrak{F}(T))^{n(\alpha)} & \xrightarrow[\mathfrak{f}_\alpha]{} & \mathfrak{F}(T)
\end{array}$$

commutes. That is, a mapping φ from A to B is an algebra homomorphism if and only if the induced mapping from $\mathfrak{F}(S)$ to $\mathfrak{F}(T)$ is a fuzzy homomorphism. That is the induced φ respects the algebraic operations $\mathfrak{f}_\alpha \in \mathcal{F}$.

3. A fuzzy equivalance relation μ on a non-empty set X is a fuzzy subset μ from XxX to I, that is reflexive ($\mu(x,x) = 1$ for all $x \in X$), symmetric ($\mu(x,y) = \mu(y,x)$ for all $x,y \in X$), and transitive ($\mu \circ \mu \leq \mu$ where $\mu \circ \mu(x,y) = \sup_{z \in X} (\mu(x,z) \wedge \mu(z,y))$). See Kaufmann [7]. Goguen [4]. Let $A = [S,F]$ be an algebra and $\mathfrak{R}(S)$ denote the set of all fuzzy relations on S. Each $f_\alpha \in F$ induces a composition of relations on $\mathfrak{R}(S)$ as follows. For each $n(\alpha)$-tuples $\mu_1, \mu_2, \ldots, \mu_{n(\alpha)} \in \mathfrak{R}(S)$

$$(f_\alpha(\mu_1,\mu_2,\ldots,\mu_{n(\alpha)}))(x,y) = \sup_{\substack{x = f_\alpha(x_1,x_2,\ldots,x_{n(\alpha)}) \\ y = f_\alpha(y_1,y_2,\ldots,y_{n(\alpha)})}} (\mu_1(x_1,y_1) \wedge \mu_2(x_2,y_2) \wedge \ldots, \mu_{n(\alpha)}(x_{n(\alpha)},y_{n(\alpha)}))$$

for all $x,y \in S$. $\mathcal{F} = (f_\alpha : f_\alpha \in F)$. If $\mathcal{E}(S)$ denotes the subset of $\mathfrak{R}(S)$ consisting of all the fuzzy equivalence relations on S, then the operations f_α on $\mathfrak{R}(S)$ defined as above restricted to $\mathcal{E}(S)$ are well-defined, thus $[\mathcal{E}(S), \mathcal{F}]$ is an algebra similar to A. A fuzzy equivalence relation $\mu \in \mathcal{E}(S)$ on an algebra $A = [S,F]$ is said to be a fuzzy congruence relation if for each $f_\alpha \in F$,

$$f_\alpha(\mu,\mu,\ldots,\mu) \leq \mu.$$

That is, a fuzzy equivalence relation is a fuzzy congruence relation if and only if it is a fuzzy subalgebra of $[\mathcal{E}(S), \mathcal{F}]$.

A fuzzy equivalence relation on X is decomposed into a class of fuzzy subsets on X in the following way. Let $\mu : X \times X \to I$ be a fuzzy equivalence relation on X. $\underset{\sim}{\omega}_1$, called the weak 1-relation on X, is a crisp equivalence relation on X defined by $x\underset{\sim}{\omega}_1 y$ if and only if $\mu(x,y) = 1$ $(x, y \in X)$. Let $\mathfrak{X} = [x]_{\underset{\sim}{\omega}}$ be the equivalence class containing x for each $x \in X$. A fuzzy subset $\tilde{\mu}_{\mathfrak{X}}$ is constructed on X as $\mu_{\mathfrak{X}}(y) = \mu(x,y)$ for all $y \in X$, x fixed in X.

Then (i) $\mu_{\mathfrak{X}}$ is well defined;

(ii) $\bigvee_{x \in X} \mu_{\mathfrak{X}} = \chi_X$.

(iii) $\mu_{\mathfrak{X}} \wedge \mu_{\mathfrak{Y}} = \mu(x,y)$ for all $x, y \in X, x \in \mathfrak{X}, y \in \mathfrak{Y}$.

(iv) $\mu_{\mathfrak{X}} \wedge \mu_{\mathfrak{Y}} = 0$ if and only if $\mu(x,y) = 0$

5. Theorem: Let A = [S,F] be a given algebra, μ a fuzzy congruence relation on S. Let $\underset{\sim}{\omega}_1$ denote as before the weak 1-relation on S. Then A is epimorphic to the quotient algebra $A/\underset{\sim}{\omega}_1$. Let T be the subclass of $\mathfrak{F}(S)$ consisting of fuzzy subsets $\{\mu_{\mathfrak{X}}\}_{x \in X}$ generated by the decomposition of the fuzzy congruence (and hence fuzzy equivalence) relation μ on S. Then [T, F] is an algebra similar to and is isomorphic with the algebra $[A/_{\underset{\sim}{\omega}_1}, F]$.

Thus the above theorem decides which subclasses of $\mathfrak{F}(S)$ form an algebra with the induced fuzzy operations.

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HYPERGROUP (II)

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ABSTRACT

Hypergroup was firstly studied in (1), which is the first step. This paper is a continuation of (1), where some examples for hypergroup are given, some properties of hypergroup are discussed, and the problem for the classification of hypergroup is considered.

1. THE BASIC THEOREMS OF HYPERGROUP

The meaning of symbols and terms used in the paper, if they are not explained specially, may be found in (1).

Suppose that G is a group from beginning to end.

Let $\mathcal{G} \subset 2^G - \emptyset$. $\mathcal{G}$ is called a hypergroup on G, if it is a group with respect to the operation:

$$AB = \{ab \mid a \in A, b \in B\} \qquad \forall A, B \in \mathcal{G}$$

that its identity element is denoted by E.

Firstly we sum up four basic theorems based on the main contents in (1).

BASIC THEOREM 1 (cardinal number theorem) If $\mathcal{G}$ is a hypergroup, then

(i) $(\forall A \in \mathcal{G})(\mathrm{card}A = \mathrm{card}E)$;

(ii) $(\forall A, B \in \mathcal{G})(A \cap B \neq \emptyset$ implies $\mathrm{card}(A \cap B) = \mathrm{card}E)$ #

BASIC THEOREM 2 (structure theorem) Let $\mathcal{G}$ be a hypergroup on G. If E is a subgroup of G, then

$$G^* = \cup\{A \mid A \in \mathcal{G}\}$$

is a subgroup of G, E is a normal subgroup of G^*, and

$$\mathcal{G} = G^*/E \qquad \#$$

BASIC THEOREM 3 (structure theorem) Let $\mathcal{G}$ be a hypergroup on G, if $e \in E$, then G^* is a subgroup of G, E is a normal subgroup of G^*, and $\mathcal{G}$ is a generalized quotient group, i.e., $\mathcal{G} = G^* | E$. #

BASIC THEOREM 4 (construction theorem) Let $E \in 2^G - \emptyset$, $E^2 = E$, and H be a subgroup of G. If H satisfies

$$(\forall x \in H)(xE = Ex)$$

then

$$\mathcal{G} = \{xE \mid x \in H\}$$

is a hypergroup on G and $H \sim \mathcal{G}$. #

These four basic theorems will play a important role in the study of hyperalgebra.

2. SOME EXAMPLES FOR HYPERGROUP

EXAMPLE 1: Let G be a positive real numbers multiplicative group. Take $E=[1,+\infty)$. Clearly E is a normal subsemigroup. Let H be a set of all positive rational numbers, which is a subgroup of G. Put

$$\mathcal{G}=\{aE \mid a\in H\}$$

From the basic theorem 4 $\mathcal{G}$ is a hypergroup on G, and E is the identity element of $\mathcal{G}$. In fact,

$$\mathcal{G}=\{[a,+\infty) \mid a\in H\}$$

and it is easy to see that $G^*=G$. From the basic theorem 3 $\mathcal{G}$ is a generalized quotient group, i.e., $\mathcal{G}=G^*|E$. Because $G=G^*$, $\mathcal{G}=G|E$. In addition, it is easy to obtain $H\cong\mathcal{G}$.

EXAMPLE 2: Let G be a real numbers additive group. Take $E=[0,+\infty)$, clearly $E+E=E$, $0\notin E$, which shows that E is not a normal subsemigroup. In addition, Z which is a set of all integers is a subgroup of G. Put

$$\mathcal{G}=\{n+E \mid n\in Z\}$$

From basic theorem 4, $\mathcal{G}$ is a hypergroup on G, and $Z\sim\mathcal{G}$, in fact $Z\cong\mathcal{G}$. A denumberable chain is formed by the elements in $\mathcal{G}$:

$$\cdots\cdots \supset(-2,+\infty)\supset(-1,+\infty)\supset E\supset(1,+\infty)\supset(2,+\infty)\supset \cdots\cdots$$

It is worth notice that the example shows there exists such a hypergroup which is not a generalized quotient group.

EXAMPLE 3: Let $(G,+,\leq)$ be a partially ordered additive group, write

$$\mathcal{G}=\{[a,b] \mid a,b\in G\}$$

where

$$[a,b]=\{c\in G \mid a\leq c\leq b\}$$

In $\mathcal{G}$ we define an algebraic operation:

$$[a_1,b_1]+[a_2,b_2]=[a_1+a_2,b_1+b_2]$$

It is easy to see that $(\mathcal{G},+)$ is a hypergroup on G and $E=\{0\}$. In addition, the mappings f and g

$$f:\ \mathcal{G}\longrightarrow G$$
$$[a,b]\longmapsto a$$
$$g:\ \mathcal{G}\longrightarrow G$$
$$[a,b]\longmapsto b$$

are all surjective homomorphisms. Clearly we have

$$Kerf=\{[0,b] \mid b\in G\}$$
$$Kerg=\{[a,0] \mid a\in G\}$$

hence

$$\mathcal{G}/Kerf\cong G\cong Kerf$$
$$\mathcal{G}/Kerg\cong G\cong Kerg$$

3. SOME PROPERTIES FOR HYPERGROUP

THEOREM 3.1 Let $\mathcal{G}\subset 2^G-\emptyset$. If $\{e\}\in\mathcal{G}$, then $\mathcal{G}$ is a hypergroup iff $G^*<G$ and $\mathcal{G}\cong G^*$.

PROOF. $\Leftarrow$: Clear

$\Rightarrow$: Clearly $E=\{e\}$, and from the theorem 2 it may be known that $\mathcal{G}=G^*/\{e\}\cong G^*$ Q.E.D.

PROPOSITION 3.1 Let $\mathcal{G}$ be a hypergroup on G. If $E<G$, then

$(\forall A\in \mathcal{G})(A\neq E \Rightarrow e\notin A)$

PROOF. Notice $\mathcal{G}=G^*/E$. Q.E.D.

THEOREM 3.2 Let G be a finite group and $\mathcal{G}\subset 2^G-\emptyset$. If $G\in\mathcal{G}$, then $\mathcal{G}$ is a hypergroup iff $\mathcal{G}=\{G\}$.

PROOF. $\Leftarrow$: Clear

$\Rightarrow$: Because $G\in\mathcal{G}$, cardG=cardE by the basic theorem 1. Since G is finite and $E<G$, G=E. So $\mathcal{G}=G/G=\{G\}$. Q.E.D.

PROPOSITION 3.2 If $\mathcal{G}$ be a hypergroup on G, then $(G,\{e\}\in\mathcal{G})\Leftrightarrow G=\{e\}$.

PROOF. $\Leftarrow$: Clear

$\Rightarrow$: Notice cardG=card{e}. Q.E.D.

THEOREM 3.3 Let G be a finite group. If $\mathcal{G}$ is hypergroup then $E=\{e\}$ iff $\mathcal{G}\cong G^*$.

PROOF. $\Rightarrow$: $E=\{e\} \Rightarrow \mathcal{G}=G^*/\{e\}\cong G^*$.

$\Leftarrow$: $\mathcal{G}\cong G^* \Rightarrow$ cardG^*=card$\mathcal{G}$=cardG^*/E=cardG^*/cardE $\Rightarrow$ cardE=1 $\Rightarrow E=\{e\}$. Q.E.D.

NOTE: In (1) we had poinied out that G is finite implies that $E<G$.

THEOREM 3.4 Let G be a finite group, and cardG be a prime number. If $\mathcal{G}$ is a hypergroup on G, then $\mathcal{G}=\{G\}$ or $\mathcal{G}=\{\{e\}\}$ or $\mathcal{G}\cong G$.

PROOF. Because $G^*<G$, $G^*=G$ or $G^*=\{e\}$.

(i) Let $G^*=G$.

(a) If $E=\{e\}$, then $\mathcal{G}=G/\{e\}\cong G$.

(b) If $E=G$, then $\mathcal{G}=G/G=\{G\}$

(ii) Let $G^*=\{e\}$.

Since $E=\{e\}$, so $\mathcal{G}=\{e\}/\{e\}=\{\{e\}\}$. Q.E.D.

PROPOSITION 3.3 If $\mathcal{G}$ be a set of all true subsemigroups in G, then $\mathcal{G}$ is a hypergroup iff $G^*<G$ and $\mathcal{G}\cong G^*$.

PROOF. $\Rightarrow$: Since $\{e\}\in\mathcal{G}$, $G^*<G$ and $\mathcal{G}=G^*/\{e\}\cong G^*$.

$\Leftarrow$: Clear Q.E.D.

PROPOSITION 3.4 If $\mathcal{G}$ be a set of all true subsemigroups contained the identity element, then $\mathcal{G}$ is a hypergroup iff card$\mathcal{G}$=1 and $(E\in\mathcal{G} \Rightarrow E^2=E)$.

PROOF. $\Rightarrow$: By $\mathcal{G}=G^*/E$ we have card$\mathcal{G}$=1, so $\mathcal{G}=\{\{e\}\}$. Clearly $\{e\}^2=\{e\}$.

$\Leftarrow$: Clear Q.E.D.

PROPOSITION 3.5 Let $\mathcal{G}$ be a set of all true subsemigroups in G. If $\mathcal{G}$ is a hypergroup, then G do not have any nontrivial subsemigroups contained the identity element, and $G^*=G$ or $G^*=\{e\}$.

PROOF. Let $H\in\mathcal{G}$ be a subsemigroup contained the identity element. It may be proved that H is a trivial subsemigroup. If this is false, then $H\neq\{e\}$, so $e\notin H$, this is a contradiction. In addition, since $G^*<G$, G^* is a subsemigroup contained the identity element, so $G^*=G$ or $G^*=\{e\}$. Q.E.D.

4. THE CLASSIFICATION ON HYPERGROUP

DEFINITION 4.1 Let $\mathcal{G}$ be a hypergroup. If there exists $H<G$ such that $H\sim\mathcal{G}$, then $\mathcal{G}$ is called Hyper 3 type; If $\mathcal{G}=G^*/E$, then $\mathcal{G}$ is called Hyper 1 type ; If $\mathcal{G}=G^*|E$, then $\mathcal{G}$ is called Hyper 2 type; If $\mathcal{G}$ is not Hyper 3, then it is called Hyper 4 type.

If quotient groups are called Hyper 0 type, then we have the following relation graph:

$$\text{Hyper } 0 \Rightarrow \text{Hyper } 1 \Rightarrow \text{Hyper } 2 \Rightarrow \text{Hyper } 3$$

The example 1 is Hyper 2 type, but not Hyper 1 type. The example 3 is Hyper 3 type, but not Hyper 2 type.

The following example will show that Hyper 1 type hypergroup may not be Hyper 0 type.

EXAMPLE 4: Let $G=Z_{12}=\{\bar{0},\bar{1},\bar{2},\dots,\overline{11}\}$. Take $E=\{\bar{0},\bar{6}\}$, $H=\{\bar{0},\bar{4},\bar{8}\}$. Write

$$\mathcal{G}=\{a+E \mid a\in H\}=\{E,\{\bar{4},\overline{10}\},\{\bar{2},\bar{8}\}\}$$

Clearly $\mathcal{G}=G^*/E$. Since $G^*=\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}\neq G$, so $\mathcal{G}$ is Hyper 1 type, but not Hyper 0 type.

AN INTERESTING OPEN PROBLEM: Whether there are Hyper 4 type hypergroup?

Now we consider the problem of classifiction for hypergroups from another voewpoint.

If put $\leqslant\ =\ \subset$, then any hypergroup on G is always a partially ordered group with respect to $\leqslant$. Particularly $\mathcal{G}$ is called a chain hypergroup when $(\mathcal{G},\leqslant)$ is a chain group. The examples which were discussed by us are almost chain hypergroups. It looks as if chain hypergroup can contain a big kind of hypergroups. This should be to specially notice.

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