A FUZZY MULTIOBJECTIVE LINEAR PROGRAMMING MODEL WITH FUZZY NUMBERS

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ABSTRACT

Fuzzy multiobjective programmings are very useful in many fields, especially in the large scale systems with several objectives, such as environmental protection, earthquake, transportation, and so on. There are two points at least to solve a fuzzy multiobjective linear programming with fuzzy numbers used as the coefficients of the constraints(FMLP). One of them is the method to transform the fuzzy linear system of constraints into a real linear one, the other is the choice of the fuzzy operators. This paper attempts to introduce a new approach based on D.Duboi's works, and to prove the property of the solution obtained from the min-bounded sum compensatory operator. In the end, a FMLP model is applied for planning the air quality in Beijing proper.

Fuzzy programming is one branch of fuzzy mathematics. The disertation written by R.E.Bellman and L.A.Zadeh in 1970 first introduced fuzzy programming, and the concept of fuzzy decision was discussed in that paper. The iteration method to solve fuzzy programmings was given and proved by H.Tanaka et al in 1973 . The concept of fuzzy Pareto solution set in multiple objects decision-making was also raised by L.A. Zadeh in 1976 . The method for solving fuzzy linear programming and fuzzy dynamic programming with fuzzy sets used as the coefficients of the constraints was proposed by C.V.Negoita in 1976 . Quite a number of specialists started the systematic research on fuzzy numbers the same year. The first thesis on fuzzy multiobjective programmings was written by H-J. Zimmermann in 1977 , he used two operators, minimum and product operators, to solve the progoammings, and proved that the solutions of the models are the efficient solutions of the previous crisp multiobjective linear programming, however, the example given by E.L.Hanan in 1978 showed that the solutions got from product operator may not be efficient ones. D.Dubois and H.Prade studied the algebraic operations of fuuzzy numbers and some operational rules in 1977 , and they introduced a new process in solving certain problems related to fuzzy linear programming with L-R fuzzy numbers used as the coefficients of the constraints in the year to come. Last year, H.Tanaka and K.Asai gave another solving process for fuzzy linear programming with fuzzy numbers used as the coefficients in both the constraints and objective functions . Based on the concept of fuzzy solution of multiobjective programmings by Feng Ying-jun, he himself and Wei Quanling went further into the relationship among the fuzzy solutions, efficient solutions and weak efficient solutions of multiobjective programming in 1981 . After that, Xu Yu approached weak efficient solution set, efficient solution set, G-efficient set, and the inclusions of the sets taken from different operators.

Seeing that there are a lot of effectual solving processes in classical mathematical programming, up to now the basic train of thought to solving fuzzy programming is to transform fuzzy programming into classical substitute one,and a solving process is considered to be completed once the transformation is realized.

It is the α-level sets used in each iteration steps that the realization of the change in the iteration method by Tamala et al simply depends on.It is also the α-level sets that the realization of the change in the process solving fuzzy programming with fuzzy set coefficients depends on,which was given by Negoita.

On the basis of the study of the L-Rtype fuzzy numbers,D.Dubois and H.Prade proposed a method to change fuzzy linear programming whose constraint coefficients are fuzzy numbers into the crisp linear programming whose constraint coefficients are real ones.The main steps of Dubois-method(D-method) are those:

In the first place,weaken the fuzzy equal constraint

$$\underset{\sim}{A}_i X = \underset{\sim}{b}_i$$

into

$$\underset{\sim}{A}_i X \subseteq \underset{\sim}{b}_i,$$

where $\underset{\sim}{A}_i=(\underset{\sim}{a}_{i1},\underset{\sim}{a}_{i2},\cdots,\underset{\sim}{a}_{im}),X=(x_1,x_2,\cdots,x_m)^T$.
Secondly,using the parameters in the L-R expression of the fuzzy numbers,

$$\underset{\sim}{A}_i=(\bar{A}_i,A_i^-,A_i^+)_{LR}$$
$$\underset{\sim}{b}_i=(\bar{b}_i,b_i^-,b_i^+)_{LR},$$

chang the constraint

$$\underset{\sim}{A}_i X \subseteq \underset{\sim}{b}_i$$

into its crisp equivalent form

$$\begin{cases} \bar{A}_i X = \bar{b}_i \\ A_i^- X = b_i^- \\ A_i^+ X \quad b_i^+. \end{cases}$$

Obviously,Negoita-method(N-method) fits alsc the fuzzy linear programming with fuzzy numbers(a special kind of fuzzy sets) as the constraint coefficients.When only two values,zero and one,is given to α,D-method becomes the particular case of N-method.Since D-method only pay attention to the particular cases,α is equal to zero or one,we need only to know the three parameters of each fuzzy numbers,the left and right spreads and the model value.Hence,we have no need to get the membership functions of every fuzzy numbers and the endpoints in the α-level sets of the fuzzy numbers,as a result, the quantity of calculation is reduced greatly.But the price to pay for the reduction is high without considering any membership functions of the fuzzy numbers,a lot of information is lost.Besides,the constraints in the substitute programming got from the both method is increased enormously in number comparing with the classical programming,so is the quantity of calculation.

There exist mainly the two methods,N-method and D-method,with which the fuzzy number linear constraint system can be transformed into a real linear one.Comparing between the two,although the quantity of calculation in D-method is much more smaller than in N-method, the possibility that the feasible solution set is a empty one has been increased,because the group of equality constraints is led into the constraint system of the substitute programming.Dubois himself gave an example in which the constraint system including two variables and two constraints is incompatible.

Therefore,whether the feasible solutuin set in the substitute programming got from D-mthod is a empty one is determined first by the state of the solutions in the group of the equality constraints,that is , no solution,unique solution or infinite groups of solutions. It is clear that it is necessary to consider the possibility of optimazation only when the third case happens.According to the theory of linear equations,when we solve a fuzzy linear programming using D-method, the neccessary condition that the feasible solution set is neither an empty one nor a singleton is that the rank number of the coefficient matrix of the equality constraits is smaller than that of the variables in the equations. It's why the linear constraint system mentioned by Dubois is incompatible that the rank of the coefficient matrix is equal to the number of the variables so that the solution of the equations is unique.

The state of the solution in equality constraints is not the only factor to decide the state of the feasible solution set,more other factors, such as the compatibility inside the inequality constraints, that between the inequality ones and equality ones,and so on.In other words,although the constraints in substitute programming by D-method are less than by N-method, the constraints are still so many and strick that the cases withought feasible solution often happened in the processes that some fuzzy programmings were solved with D-method by the auther.Now,the problem is"Can a new approach be introduced to transform the system of fuzzy number linear constraints into the system of real linear ones by means of some basic idears and symboles in D-method,so that the constraints in the substitute problem will be fewer and looser to meet the needs of some properties in fuzzy set theory at the same time?"

The basic theoretical foundation of D-method is Zadeh's inclusion, i.e.,

$$\underset{\sim}{A} \subseteq \underset{\sim}{B}$$

iff $\forall x \in X$,

$$\underset{\sim}{A}(x) \leq \underset{\sim}{B}(x),$$

and

$$A=B$$

iff

$$\underset{\sim}{A} \subseteq \underset{\sim}{B} \text{ and } \underset{\sim}{A} \supseteq \underset{\sim}{B}.$$

From this definition we know that

$$\underset{\sim}{A}_i X = \underset{\sim}{b}_i$$

can be weakened into either

$$\underset{\sim}{A}_i X \subseteq \underset{\sim}{b}_i$$

or

$$\underset{\sim}{A}_i X \supseteq b_i.$$

Moreover,the so called"equivalwnt form" is not really equivalent to the fuzzy linear constraint system in Zadeh's sence.The case in fig.1 dose not satisfy that

$$\underset{\sim}{A}_i X \subseteq \underset{\sim}{b}_i$$

in Zadeh's sence,but satisfies the "equivalent form".

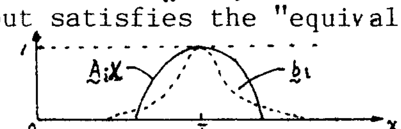

Fig.1 An example for the non-equivalent case in D-method

This example shows that

$$\underset{\sim}{A}_i X \subseteq \underset{\sim}{b}_i$$

is only the sufficient condition for the "equivalent form",but not the necessary one.

The new approach expounded in this paper depends on the definition[12] of weak inclusion.

Definition:x α-belongs to $\underset{\sim}{A}$,iff $x\in \underset{\sim}{A}_\alpha$;$\underset{\sim}{A}$ is weakly included in $\underset{\sim}{B}$,denoted $A \prec_\alpha B$,as soon as all the elements of X α-belong to $\bar{\underset{\sim}{A}}$ or to $\underset{\sim}{B}$;mathematically,

$$A \prec_\alpha B$$

iff

$$x\in(\bar{\underset{\sim}{A}}\cup\underset{\sim}{B})_\alpha \qquad \forall x\in X.$$

In the light of the definition,we have following results.

Proposition 1:Given two L-R fuzzy numbers,

$$\underset{\sim}{A}=(\bar{A},A^-,A^+)_{LR},$$
$$\underset{\sim}{B}=(\bar{B},B^-,B^+)_{LR},$$
$$\underset{\sim}{A}(x)=\begin{cases} L_A((\bar{A}-x)/A^-) & x\leq\bar{A},A^->0,\\ R_A((x-\bar{A})/A^+) & x\geq\bar{A},A^+>0;\end{cases}$$
$$\underset{\sim}{B}(x)=\begin{cases} L_B((\bar{B}-x)/B^-) & x\leq\bar{B},B^->0,\\ R_B((x-\bar{B})/B^+) & x\geq\bar{B},B^+>0,\\ 0 & \text{otherwise};\end{cases}$$

where L_A,L_B,$R_A$and $R_B$are all strictly monotone decreasing,continuons mapping from $[0,+\infty)$ to $[0,1]$,B^- and B^+ are non-zero meanwhile,if

$$\begin{cases}\bar{A}\geq\bar{B}\\ \bar{A}-A^-<\bar{B},\end{cases}$$

(1)if B^- is equal to null, then

$$\underset{\sim}{B}\subseteq\underset{\sim}{A}$$

in $(-\infty\ ,\ \bar{B})$ in Zadeh's sence,and

$$B\prec_\alpha A$$

in $[B,+\infty)$;

(2)if $B^+$is equal to zero,then

$$B\prec_\alpha A$$

in $(-\infty,\bar{B})$,and

$$\underset{\sim}{B}\subseteq\underset{\sim}{A}$$

in $[B,+\infty)$ in Zaedh's sence;

(3)if neither B^- or B^+ is zero,then

$$B\prec_\alpha A$$

in $(-\infty,+\infty)$.

The proving process is omitted because it is too long.

Corollary 1:On the assumptions in proposition 1 ,if

$$\begin{cases}\bar{A}\geq\bar{B}\\ \bar{A}-A^-<\bar{B}-B^-,\end{cases}$$

all of the conclusions in proposition 1 remain the same.

It is proved that when that

$$\underset{\sim}{A}_i X=\underset{\sim}{b}_i$$

is weakened into that

$$\underset{\sim}{A}_i X\supseteq\underset{\sim}{b}_i,$$

the real linear system of consteaints

$$\begin{cases}\bar{A}_i X\geq\bar{b}_i\\ (\bar{A}_i-A_i^-)X<\bar{b}_i-b_i^-,\end{cases}$$

is a sufficient condition that $\underset{\sim}{b}_i$ is included in $\underset{\sim}{A}_i X$ in weak inclusion sence at least, On the other hand, when that

$$\underset{\sim}{A}X=\underset{\sim}{b}_i$$

is weakened into that

$$\underset{\sim}{A}_i X\subseteq \underset{\sim}{b}_i,$$

the real linear system of constraints,

$$\begin{cases}\bar{A}_i X\leq \bar{b}_i\\ (\bar{A}_i - A_i^-)X > \bar{b}_i - b_i^-,\end{cases}$$

is a sufficient condition that $\underset{\sim}{b}_i$ is included in $\underset{\sim}{A}_i X$ in weak inclusion sence at least.

Because this approach is based on the definition of weak inclusion, it is called as weak-inclusion approach. Comparing with D-method, the approach reduces the constraints in number by one-third, the constraint conditions are loosed notably, and the quantity of the calculation and the possibility that the feasible solution set is empty has been decreased remarkably. The fuzzy number linear system of constraints is compatible under weak-inclusion approach, which was incompatible under D-method as mentioned.

The other point in fuzzy multiobjective programming is about the choice of fuzzy operators for the aggragation of the fuzzy sets in the programming.

The widely applications of fuzzy set theory in decision-making provide specialists with the facts that the operators to aggragate fuzzy sets should be even more close to the process of human decision-making itself. People used to employ minmum and product operators for the aggragations, but many papers showed that if there exists one i so that

$$\mu_{\underset{\sim}{f}_i}(x)<\mu_{\underset{\sim}{f}_j}(x) \qquad \forall x\in X, \forall j,\ j\neq i,$$

then

$$\min_k(\mu_{\underset{\sim}{f}_k}(x))\equiv\mu_{\underset{\sim}{f}_i}(x) \qquad \forall x\in X.$$

It is clear that such a result isn't in conformity with human decision-making. Later, the concept of compensation was proposed. This concept means that the minmalgrade of membership should not be used as the unique standard in making decision, both maxmum and minmum membership values should be in overall consideration and compensate each other. The aggragation between two fuzzy sets, $\underset{\sim}{A}$ and $\underset{\sim}{B}$, using compensatary operator is expressed as follows,

$$\mu_{\underset{\sim}{A}\theta\underset{\sim}{B}}=\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}^{\gamma}\cdot\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}^{1-\gamma}$$

or

$$\mu_{\underset{\sim}{A}\theta\underset{\sim}{B}}=\gamma\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}+(1-\gamma)\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}},$$

where the parameter γ which is called as the grade of compensation is a constant between zero and one, and dual operators are usually used for the operations of set union and intersection.

M.K.Luhandjula demonstrated a compensatory operator used in multiobjective programming, he emploied the non-dual operators to define the operations of union and intersection in order to remain the linear proprety. Using the min-bounded sum(MBS) operator he raised, the expression is that

$$\mu_{\underset{\sim}{A}\theta\underset{\sim}{B}}=\gamma\min(\mu_{\underset{\sim}{A}},\mu_{\underset{\sim}{B}})+(1-\gamma)\min(1,\mu_{\underset{\sim}{A}}+\mu_{\underset{\sim}{B}}).$$

He proved that the substitute programming obtained from MBS operator can be solved by simplex. But Luhandjula pointed out only that the solutions of the substitute programming is not the efficient one of the previous multiobjective programming,he did not explore whether the solution is weakly efficient.However,if the solution is not even a weak efficient one, the operator has little practical value to solve multiobjective programming. Hence,the property of the solution is discussed in the following proposition.

Proposition 2 : The result obtained from MBS operator is a weak efficient solution.

Proof:Given a arbitrary positive integer N,$\forall x_1, x_2 \in X$,if $\forall\ 1 \le i \le N$

$$\mu_i(x_2) < \mu_i(x_1),$$

then for

$$\mu_{I_1}(x_1) = \min_i \mu_i(x_1),$$
$$\mu_{I_2}(x_2) = \min_i \mu_i(x_2),$$

when $I_1 = I_2$,from the known condition,it holds that

$$\mu_{I_2}(x_2) < \mu_{I_1}(x_1);$$

otherwise,it holds that

$$\mu_{I_2}(x_2) \le \mu_{I_1}(x_2) < \mu_{I_1}(x_1),$$

that is,

$$\min_i \mu_i(x_2) < \min_i \mu_i(x_1).$$

It is also true in the same reason that

$$\sum_i \mu_i(x\ ) < \sum_i \mu_i(x_1),$$

i.e.,

$$\min[1, \sum_i \mu_i(x\ )] \le \min[1, \sum_i \mu_i(x\ )].$$

therefore, $\forall \nu \in (0,1)$,

$$\begin{aligned} & h[\mu_1(x_2), \cdots, \mu_N(x_2)] \\ &= \nu \min_{1 \le i \le N} \mu_i(x_2) + (1-\nu)[1, \sum^N \mu_i(x_2)] \\ &< \nu \min_{1 \le i \le N} \mu_i(x_1) + (1-\nu)[1, \sum^N \mu_i(x_1)] \\ &= h[\mu_1(x_1), \cdots, \mu_N(x_1)], \end{aligned}$$

this means that MBS operator is monotone increasing.

In accordance with the proved conclusions ,if a operator

$$h:(0,1]^N \longrightarrow [0,1]$$

monotone increasing,i.e.,$\forall x_1, x_2 \in (0,1]^N$,when $x_1 < x_2$

$$h(x_1) < h(x_2),$$

then the solution obtained from this operator is a weak efficient one of the provious multiple objects programming.Terefore,the solution obtained from MBS operator is a weak efficient one.

Fuzzy multiobjective programming is very useful for system analysis. A FMLP model for planning air quality in Beijing proper is introduced here.

$$(\text{FMLP}): \begin{cases} \min(\sum_k \sum_j x_{jk}\ ,\ \sum_k \sum_j c_k EQ_{jk}\ x_{jk}) \\ \sum_k \sum_j \underset{\sim}{a}_{ijk}\ x_{jk} \supseteq \underset{\sim}{b}_i & \forall\ i \\ 0 \le x_{jk} \le 1 \end{cases}$$

where x_{jk} is the elimination rate of pollutant emitted from kth class of sources in jth source area;c_k is the cost of the control measures adopted to kth class of source;EQ_{jk} is the quantity of pollutant emitted from kth class of sources in jth source area; $\underset{\sim}{a}_{ijk}$ is a L-R fuzzy numger,whose model value is the diffusion simulation value of the contribution of kth class of sources in jth source area to ith source area,and the determinations of the left and right spreads depends on the measurement values;$\underset{\sim}{b}_i$ is also a L-R fuzzy number,whose model value is the difference between the simulation value in ith source area and environmental standard and the determinations of left and right spreads depends on the measurement values.

The weak inclusion approach is emploied to turn the fuzzy number linear system of constraints in FMLP into a real linear one, the MBS operator is used to get the last substitute programming,and the weak efficient solution is obtained in the end.

From the results it is known that the tatol of the reduction rates, the first object in FMLP,decreases by 27 per cent and the tatol of the control cost,the second object in the model, reduces by more than 3 million yuan a year as compare with the totals obtained from the liear programming model often applied in the regulation of tatol emission in Japen.

Conclusion

This paper attempts to provide a new way for solving fuzzy multi-objective linear programming with fuzzy numbers as the coefficients of the cinstraints,and apply the model in decision-making,planning regional air quality.

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