# CONTRIBUTION TO A MATHEMATICAL THEORY OF FUZZY GAMES

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ABSTRACT.

The theory of games and its applications in economics is valid for a certain kind of precise or accurate conditions. If we postulate some imprecision in the usual models, we have to transform the terms of the problem to insert a discourse on the imprecision.
The subject of this paper is to permit the exhibition of a f-continuous utility on the strategy-set of a player. If we are in the case of players constrained to play together in a simultaneous stroke, or in the case where each gambler doesn't know the other gambler's payoffs, the preference is clearly lexicographical and so the pre-order is not continuous. To overshoot this difficulty, we have to use a new fuzzy-approach in three levels:
- first, we make a hypothesis on the player's distinction between two strategies which are extremely neighbouring, called the hypothesis of no-local-distinction.
- second, we prove the existence of a new continuous function on the strategy-set, in a case of a fuzzy continuous lexicographical preorder.
- third, we try to rediscover some classical results of the theory of games and we show that results issued from the usual way are just the limit case of ours.

Keywords : Socio-economic systems, Decision-Making, Fuzzy Programming.

## 1. Introduction.

1.1. When we consider all the games with the following structure -gamblers constrained to play in a simultaneous stroke, gamblers with imperfect information on the payoffs- the classical theory of games without cooperation gives us some precise results. ( Hervé MOULIN, 1981 ). The fundamental result is about the max-min criterium. This criterium, in fact, sets up a tie between advisable strategies and optimal ones. In the case of that games, we try to exhibit a strategical utility function with one argument, continuous on the strategy set of each player.

The difficulty was to overshoot the problem of no-continuity of the lexicographical fuzzy preorder. This is possible with fuzzy mathematics and a hypothesis which economic meaning is not so restrictive.

1.2. We consider a game without cooperation called nocooperative game with n-gamblers. Each player i ( i ;1,......,n ) has a continuous strategy set $\underline{X}_i$ and a function of payoff denoted $P_i$, defined on the $\prod_{i=1}^{n} \underline{X}_i$ set. We assimilate $\underline{X}_i$ to a part of $\mathbb{R}$ and the payoffs are supposed to be continuous on it. The game is shown like that: ( $\underline{X}_1$, $\underline{X}_2$, ....., $\underline{X}_n$; $P_1$, $P_2$, ....., $P_n$ ). For using a lexicographical preorder on the different strategies, we have to assimilate now every strategy with a vector of associated payoffs. In other words, $x_i \in \underline{X}_i$ ; $x_i = \vec{x}_i$ : ( $P_i(x_i, x_{\hat{j}}), \ldots\ldots P_i(x_i, x_{\hat{j}'})$, with $(x_{\hat{j}}, \ldots\ldots\ldots, x_{\hat{j}'})$ the (n-1)-tuples of strategies used by the other gamblers than i.

Then, we consider an application t, which permits to arrange its co-ordinates; $\vec{x}_i \longrightarrow t(\vec{x}_i) = \vec{x}_i^{\,+}$ with $\vec{x}_i^{\,+}$ a vector with arranged constituents in a growing way, i.e;$(x_i^o \geqslant x_i^p$ if $o > p)$.

1.3. Now, we define a lexicographical preorder on $\underline{X}_i$, $\underline{X}_i \subset \mathbb{R}^m$, m being the number of strategies and so the cardinal of $\underline{X}_i$.

$$\vec{x}_i^{\,+} \succ_l \vec{x}_j^{\,+} \text{ iff } \exists t_o \in \{1,\ldots\ldots,m\} \; ; \; \forall t < t_o \begin{cases} P_i^t(x_i, x_{\hat{j}}) = P_i^t(x_j, x_{\hat{j}}) \\ P_i^{t_o}(x_i, x_{\hat{j}}) > P_i^{t_o}(x_j, x_{\hat{j}}) \end{cases}$$

Now, we shall write $\vec{x}_i^{\,+}$ in that way : $\mathbf{x}_i$.

The usual lexicographical preorder $\succ_l$ is reflexive, transitive and total. But it is not a continuous preorder because of its topological properties. (Chang C.L , 1968 ). To overshoot this difficulty we have to use a fuzzy preorder and a hypothesis of no-local-distinction.

## 2. The analysis framework.

2.1. Let H, a binary fuzzy relation between the elements of $\underline{X}_i^2$ ; $x_i H x_j = \{(x_i,x_j),\ h; \forall x_i \in \underline{X}_i,\ \forall x_j \in \underline{X}_i,\ h(x_i,x_j) \in \underline{M}\}$ where $h(x_i,x_j)$ denotes the level of preference or indifference between the two strategies and $\underline{M}$ is $[0,1]$. ( Ponsard 1981 , Zadeh 1965)

2.2. The hypothesis of no-local-distinction.
$\forall x_i \in \underline{X}_i$ ; $\forall V(x_i) \in \vartheta(x_i)$ ( $V(x_i)$ a neighbourhood of $x_i$, element of $\vartheta(x_i)$ set of the neighbourhoods of $x_i$), $\exists x_j \in \underline{X}_i \cap V(x_i)$ such that $h(x_i,x_j) = h(x_j,x_i)$. Now, we can write :

if $x_i \succ_1 x_j$ then $h(x_i,x_j) \geqslant h(x_j,x_i)$.

2.3. The properties of H.
The fuzzy binary relation H is reflexive : $\forall x_i \in \underline{X}_i$, $h(x_i,x_i) = 1$, transitive : $\forall (x_i,x_j,x_k) \in \underline{X}_i^3$, $h(x_i,x_k) = \underset{x_j}{\text{Max}} \left[\text{Min}(h(x_i,x_j),h(x_j,x_k))\right]$, and complete : $\forall (x_i,x_j) \in \underline{X}_i^2$ , $h(x_i,x_j) \gtrless h(x_j,x_i)$.

2.4. The fuzzy subset $X_i \subset \underline{X}_i$
With the fuzzy preorder H, we define a fuzzy subset $X_i$ like that:
$\forall x_i \in \underline{X}_i$ ; $\mu_{X_i}(x_i) = \text{Min}\left[h(x_i,x_j) \text{ such that } h(x_i,x_j) = h(x_j,x_i)\right]$.
The membership function makes a tie between the different indifference areas with ; $\mu_{X_i}(x_i) \geqslant \mu_{X_i}(x_j)$ iff $h(x_i,x_j) \geqslant h(x_j,x_i)$.

2.5. Definition.
A fuzzy subset $X_i$ is said connected iff : $\forall (X_1,X_2) \subset X_i^2$, $(X_1,X_2)$ non-empty, closed and disjoined, then $\exists x_i \in X_i$ such that :
$\mu_{X_i}(x_i) \neq \text{Max}\left[\mu_{X_1}(x_i), \mu_{X_2}(x_i)\right]$.

2.6. Proposition 1.

Let $X_i$ a fuzzy subset (fss) of $\mathbb{R}^m$, then it exists a fss D, $D \subset X_i$ such that $\bar{D} = X_i$ ($\bar{D}$ the closure of D) and suppD is countable.

Proof : $X_i \subset \mathbb{R}^m$ : $\mathbb{R}^m$ is the referential set of $X_i$. We know in order with the theorem of decomposition ( Kaufmann, 1977)

that $X_i = \underset{\alpha}{\mathrm{Max}}[\alpha \ X_{\underline{i}}]$, with $\vec{\alpha}$ a real vector composed of each $\alpha$, and $X_{\underline{i}} \overset{\alpha}{\subset} \mathbb{R}^m$. ( $0 < \alpha \leqslant 1$). $X_{\underline{i}}$ is included in $\mathbb{R}^m$ and so it admits an usual subset $\underline{D}_\alpha$ (countable) such that $\overline{\underline{D}}_\alpha = X_{\underline{i}}$ (Debreu, 1970 ). For each $\alpha$ we can associate a couple $(X_{\underline{i}}, \underline{D}_\alpha)$, which has the following properties: $\bigcup_\alpha \underline{D}_\alpha = D$ and $\overline{D} = \underline{X}_i$ because the closure of a union is the union of the closures. Now, we can consider that : $\alpha \overline{\underline{D}}_\alpha = \alpha X_{\underline{i}}$ for each $\alpha$ and so $\mathrm{Max}\,\alpha\, \overline{\underline{D}}_\alpha = \mathrm{Max}\,\alpha X_{\underline{i}}$.
We have just now to conclude : $X_i = \mathrm{Max}\,\alpha X_{\underline{i}} \overset{\alpha}{=} \mathrm{Max}\,\alpha \overline{\underline{D}}_\alpha = \overline{D}$ with $\bigcup_\alpha \underline{D}_\alpha = \underline{\mathrm{supp} D}$.

Q.E.D.

2.7. Proposition 2.

$\forall (x_i, x_j) \in \underline{X}_i^2$ ; if $\mu_D(x_i) \leqslant \mu_D(x_j)$ then $\mu_{X_i}(x_i) \leqslant \mu_{X_i}(x_j)$.

Proof : The closure of D, $\overline{D}$ is such that if $\mu_D(x_i) > 0$ then $\mu_D(x_i) = \mu_{\overline{D}}(x_i)$. If $\mu_D(x_i) \leqslant \mu_D(x_j)$ then $\mu_{\overline{D}}(x_i) \leqslant \mu_{\overline{D}}(x_j)$.
Now $\overline{D} = X_i$ then $\mu_{\overline{D}}(x_i) = \mu_{X_i}(x_i) \leqslant \mu_{\overline{D}}(x_j) = \ X_i(x_j)$.

Q.E.D.

## 3. The continuity of preferences.

3.1. $\forall\ x_i \in \underline{X}_i$, the following subsets : $\{x_j \in X_i$ such that $\mu_{X_i}(x_i) \geqslant \mu_{X_i}(x_j)\}$ and $\{x_j \in X_i$ such that $\mu_{X_i}(x_i) \leqslant \mu_{X_i}(x_j)\}$ are closed. (rem)

3.2. Proposition 3.

Let $(x_j, x_j) \in X_j^2$, $X_j$ connected fss with: $\mu_{X_i}(x_i) < \mu_{X_i}(x_j)$ then it exists $x_k$ , $x_k \in D$, $D \subset X_i$ such that: $\mu_{X_i}(x_i) < \mu_{X_i}(x_k) < \mu_{X_i}(x_j)$.

rem : $X_i^{x_i} = \{x_j \in X_i$ such that $\mu_{X_i}(x_j) \leqslant \mu_{X_i}(x_i)\}$ ; if $x_j \in X_i^{x_j}$ then $\mu_{X_i^{x_i}}(x_j) = \mu_{X_i}(x_j)$ and if $x_j \notin X_i^{x_j}$ then $\mu_{X_i^{x_i}}(x_j) = 0$ .

Proof : We consider the two following fss;
$X^i = \{x \in X_i$ such that $\mu_{X_i}(x) \leqslant \mu_{X_i}(x_i)\}$ and
$X^j = \{x \in X_i$ such that $\mu_{X_i}(x) \geqslant \mu_{X_i}(x_j)\}$ . $X^i$ and $X^j$ are disjoined, by definition, non-empty and with the hypothesis of continuity of preferences, closed in $X_i$. We have assumed that $X_i$ is a connected fss. It means: $\exists\, x \in X_i$ such that $\mu_{X_i}(x) \neq \max\left[\mu_X i(x), \mu_X j(x)\right]$.
We suppose now the opposed-proposition : $\nexists\ x_k \in D$ such that $\mu_{X_i}(x_i) < \mu_{X_i}(x_k) < \mu_{X_i}(x_j)$. We can write that, in an other way: if $\nexists\ x_k$ such that $\mu_{X_i}(x_i) < \mu_{X_i}(x_k) < \mu_{X_i}(x_j)$, then $\mu_D(x_k) = 0$. So, $\mu_D(x_k) \neq 0$ is equivalent to $\mu_{X_i}(x_i) \geqslant \mu_{X_i}(x_k)$ or $\mu_{X_i}(x_k) \geqslant \mu_{X_i}(x_i)$. It means that $\mu_D(x_k) \neq 0$ iff $x_k \in X^i \cup X^j$. Now, we know that $D \subset X_i$ and $\mu_D(x_k) \leqslant \mu_{X_i}(x_k)$, $\forall\, x_k \in D$. Now, $\mu_D(x_k) \neq 0$ iff $\mu_D(x_k) \leqslant \max\left[\mu_X i(x_k), \mu_X j(x_k)\right]$.
Furthermore, we know that $\bar{D} = X_i$ so:
$\mu_{\bar{D}}(x_k) = \mu_{X_i}(x_k)$ , $\forall\, x_k \in D$. We have $D \subset X^i \cup X^j$ so $\bar{D} \subseteq X^i \cup X^j$, (union of closed). Then, we can write :
$\mu_{X_i}(x_k) \leqslant \max\left[\mu_X i(x_k), \mu_X j(x_k)\right]$ . This is only possible with (see rem p.4) $\mu_{X_i}(x_k) = \max\left[\mu_X i(x\ ), \mu_X j(x_k)\right]$, $\forall\, x_k \in X_i$. But $X_i$ is connected, so there is a contradiction and the opposed-proposition is not valid.

Q.E.D.

3.3. Proposition 4.

Let a fss D, whose suppD is countable. Then it exists an utility function on suppD.

Proof : The proof is obvious when the structure of the non-fuzzy set is countable, and this is the case of suppD. (see Gérard-Varet, Thisse, Prêvot 1976 and Ponsard 1981 ).

Q.E.D.

3.4. This utility function keeps off the fuzzy preorder and can be obtained like that : ds : strategical utility on suppD;
$\forall\, x_i \in D$ then $ds(x_i) = \mu_D(x_i)$.

3.5. Proposition 5.

Let ds the utility function defined on suppD and $D^i = \{ x \in D \text{ such that } \mu_D(x) \leq \mu_D(x_i) \}$
$D_i = \{ x \in D \text{ such that } \mu_D(x) \geq \mu_D(x_i) \}$,
then $\sup ds(D^i) = \inf ds(D_i)$.

Proof :

$\mathrm{Supp}D^i \cap \mathrm{supp}D_i = x_i$. We know that $ds(x_i) = \mu_D(x_i)$ so $\sup ds(D^i) = \sup \mu_{D^i}(x)$ . Then $\sup ds(D^i) = \mu_D(x_i)$. We could do the same thing for $\inf ds(D_i) = \mu_D(x_i)$.

Q.E.D.

3.6. We decide to call $fu(x_i)$ the common value of the two fss, $D^i$ and $D_i$. Now, $ds(x_i) = \mu_D(x_i) = fu(x_i)$. With the second proposition (see p.4) we have $\mu_D(x_i) = \mu_{X_i}(x_i)$ and so $fu(x_i) = \mu_{X_i}(x_i)$. In that way, we make the extension from suppD to $\mathrm{supp}X_i = \underline{X}_i$. ( with $fu(x) = 0$ iff $x \notin \mathrm{supp}X_i$).

4. The continuity of fu.

THEOREM I : Let $X_i$ a connected fuzzy subset of $\mathbb{R}^m$, totally pre-ordered by H. Under the hypothesis of continuity of preferences, it exists a strategical utility function fu, continuous on $\underline{X}_i$, with $fu(x) = \mu_{X_i}(x), \forall x \in \underline{X}_i$. (And $X_i$ the associated fss, and H the lexicographical fuzzy preorder.)

Proof :

We are going to show that for c, $c \in ]0,1[$, the inverse image of the closed semi-straight-line $[c,1]$ is a closed subset in $\mathrm{supp}X_i$, by fu. We could do the same proof, in the case of the other closed semi-straight-line $[0,c]$ .

Let t; $0 < t < 1$ :

$X_t = \{ x \in \underline{X}_i \text{ such that } fu(x) \leq t \}$,

$X^t = \{ x \in \underline{X}_i \text{ such that } fu(x) \geq t \}$ . $[c,1] = \bigcap_{\substack{r<c \\ r \in \underline{N}}} [r,1]$

with $\underline{N} = \{ r \in [0,1] \ / \exists x_i \in D;\ \mu_D(x_i) = r \}$

Now, $fu^{-1}([c,1]) = \bigcap_{\substack{r \leqslant c \\ r \in \underline{N}}} fu^{-1}([r,1])$.

$\underline{suppX^c} = \bigcap_{\substack{r \leqslant c \\ r \in \underline{N}}} \underline{suppX^r}$. We know that, if $X^c$ is closed (resp open) then $\underline{suppX^c}$ is closed (resp open).

So, $X^c = \bigcap_{\substack{r \leqslant c \\ r \in \underline{N}}} X^r$.

Let $x_i \in X_i$ such that $\mu_{X_i}(x_i) = r$;

$X^r = \{ x_j \in X_i$ such that $\mu_{X_i}(x_i) \leqslant \mu_{X_i}(x_j) \}$ which is a closed subset with the hypothesis of continuity of preferences. A countable intersection of closed fss being a closed fss, $X^c$ is a closed fss and so $\underline{suppX^c}$ too .

Q.E.D.

The proof of this theorem is possible because of the first hypothesis of no-local-distinction. It means that a gambler is indifferent between a strategy and one of its neighbours, because he is not able to distinguish them -he doesn't want to distinguish them. If we are in a case of a continuous set of strategies, this hypothesis is simply, the description of an imprecision of prefe-rence. The acuity of preference of a human-gambling is such that it doesn't permit to distinguish some infinitesimal variations of payoffs, for example.It means that a gambler is indifferent bet-ween two vectors whose mathematical "norm" is identical. (It does not belong to us to fix a distinction-level, this one is own's gambler.)

## 5. Fuzzy saddle-points set.

5.1. Definition.
Let a strategy $x \in \underline{X}_i$ . x is called f-no-dominated iff $\exists\, x_i \in \underline{NDX}_i$ such that $fu(x) = fu(x_i)$. ( $\underline{NDX}_i$ the set of no-dominated strategies of $\underline{X}_i$ and $\underline{DX}_i$ the set of dominated strategies.)

5.2. Proposition 6.

$$\forall x_i \in \underline{DX}_i, \exists\, x_j \in \underline{X}_i \text{ such that } fu(x_i) \leqslant fu(x_j).$$

And let us assume $\underline{X}_i$ compact.

Proof : $\forall x_i \in \underline{DX}_i \ \exists x_j \in \underline{X}_i$ such that $x_i$ is dominated by $x_j$, because $\underline{X}_i$ is compact and $P_i$ continuous. Then $\forall x_g \in \underline{X}_g$, $P_i(x_i,x_g) \leqslant P_i(x_j,x_g)$. And $\exists x_g \in \underline{X}_g$ such that $P_i(x_i,x_g) < P_i(x_j,x_g)$ so, $\min P_i(x_i,x_g) \leqslant \min P_i(x_j,x_g)$ then $x_i \prec_1 x_j$, so $\mu_{X_i}(x_i) \leqslant \mu_{X_i}(x_j)$ and $fu(x_i) \leqslant fu(x_j)$.

Q.E.D.

THEOREM II : For $\underline{X}_i$ compact and $P_i$ continuous, the strategy $\bar{x}$ such that, $\bar{x} \in \underline{X}_i$, $fu(\bar{x}) = \max\limits_{\underline{X}_i} fu(x_i)$ is f-no-dominated.

Proof : We consider a strategy $x_j$ such that $x_j$ dominates $\bar{x}$. With the proposition 6 (see p.7) it means that $fu(x_j) \geqslant fu(\bar{x})$. But $\bar{x} = \mathrm{Arg}\left\{\max\limits_{\underline{X}_i} fu(x_i)\right\}$ so $fu(x_j) = fu(\bar{x})$. By the definition p.7, $\bar{x}$ is f-no-dominated. (rem)

Q.E.D.

THEOREM III : In a zero-sum two-person game $(\underline{X}, \underline{Y}, P, -P)$, with $\underline{X}$ and $\underline{Y}$ compact sets and $P$ continuous, the fuzzy saddle-points set $\left\{\mathrm{Arg}\left\{\max\limits_{\underline{X}} fu_x(x)\right\}, \mathrm{Arg}\left\{\max\limits_{\underline{Y}} fu_y(y)\right\}\right\}$ contains the usual saddle-point of this game.

Proof : If $\bar{x} \in \mathrm{Arg}\left\{\max\limits_{\underline{X}} fu_x(x)\right\}$ then $fu_x(\bar{x}) \geqslant fu_x(x)$, $\forall x \in \underline{X}$ so $\mu_X(\bar{x}) \geqslant \mu_X(x)$.

It means : $\min p(\bar{x},y) \geqslant \min p(x,y)$, $\forall x \in \underline{X}$, $\forall y \in \underline{Y}$. Then, $\min p(x,y) = \max -p(x,y)$ and so, $\min p(\bar{x},y) \geqslant \max -p(x,y)$, that, for all $x \in \underline{X}$, for all $y \in \underline{Y}$ . So, we can write :

$$p(\bar{x},y) \geqslant -p(x,y);\ \forall x \in \underline{X},\ \forall y \in \underline{Y} . \qquad \text{(a)}$$

If $\bar{y} \in \mathrm{Arg}\left\{\max\limits_{\underline{Y}} fu_y(y)\right\}$ then $fu_y(\bar{y}) \geqslant fu_y(y)$; $\forall y \in \underline{Y}$

rem : $x_i \in \underline{NDX}_i \Rightarrow x_i \in f\text{-}\underline{ndx}_i \Leftrightarrow \underline{NDX}_i \subseteq f\text{-}\underline{ndx}_i$. The criterium of f-no-domination is a criterium which is weaker than usual one.

so $\mu_Y(\bar{y}) \geqslant \mu_Y(y)$.

It means : $\min -p(x,\bar{y}) \geqslant \min -p(x,y), \forall x \in \underline{X}, \forall y \in \underline{Y}$. Then; $\min -p(x,y) = \max p(x,y)$ and so, $\min -p(x,\bar{y}) \geqslant \max p(x,y)$, that, for all $x \in \underline{X}$, for all $y \in \underline{Y}$. So, we can write :

$$-p(x,\bar{y}) \geqslant p(x,y); \forall x \in \underline{X}, \forall y \in \underline{Y}. \qquad \text{(b)}$$

The two propositions (a) and (b) permit to write:

$$p(x,\bar{y}) \leqslant -p(x,y) \leqslant p(\bar{x},y) \text{ and so } p(x,\bar{y}) \leqslant p(\bar{x},y) \qquad \text{(c)}$$

We know that $\underline{X}$ is a compact and p continuous, so $p(.,y)$ reaches its maximum on $\underline{X}$ : $\exists \hat{x} \in \underline{X}$ such that $p(x,\bar{y}) \leqslant p(\hat{x},\bar{y})$, and that for all $x \in \underline{X}$, and so for $\bar{x}$. Thus,

$$p(\bar{x},\bar{y}) \leqslant p(\hat{x},\bar{y}) \qquad \text{(d)}$$

We know that $\underline{Y}$ is a compact and p continuous, so $p(x,.)$ reaches its maximum on $\underline{Y}$ : $\exists \hat{y} \in \underline{Y}$ such that $p(\bar{x},\hat{y}) \leqslant p(\bar{x},y)$, and that for all $y \in \underline{Y}$, and so for $\bar{y}$. Thus,

$$p(\bar{x},\hat{y}) \leqslant p(\bar{x},\bar{y}) \qquad \text{(e)}$$

In order with (c), i can write:
$p(\hat{x},\bar{y}) \leqslant p(\bar{x},\hat{y})$ so, with (d) and (e):
$p(\bar{x},\bar{y}) \leqslant p(\hat{x},\bar{y}) \leqslant p(\bar{x},\hat{y}) \leqslant p(\bar{x},\bar{y})$ then,
$p(\hat{x},\bar{y}) = p(\bar{x},\hat{y}) = p(\bar{x},\bar{y})$. So, we can conclude: $(\hat{x},\hat{y}) = (\bar{x},\bar{y})$.
Now, we can rewrite(c), (d), (e) to obtain the usual condition of a saddle-point:

$$p(x,\bar{y}) \leqslant p(\bar{x},\bar{y}) \leqslant p(\bar{x},y) .$$

Q.E.D.

This theorem is a fuzzy generalisation of Von Neumann and Morgenstern's one. It permits for the number of equilibria to grow bigger, and with the first theorem on the continuity of the fuzzy strategical utility, it permits to envisage the reach of the game-equilibria like a simple research of a supremum. It is obvious that, the fuzzy generalisation of Nash equilibrium will be independent of Kakutani's point-to-set theorem because of that one-argument strategical utility.

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