EVALUATOR, DISTANCE, SIMILARITY DEGREE

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1. INTRODUCTION

Similarity degree of fuzzy set, which be first given by Wang Peizhuang, is an important concept in fuzzy theory. It plays an important role in fuzzy clustering and classification, so it is unceasingly researched, and many people try to give various similarity degree [1]. But at present given kinds of it are only a few, therefore cannot satisfy the need for application. In this paper we discuss the relations among Evaluator, Distance and Similarity degree, and give the methods to construct them, such that we possibly have effective methods, of which we can make use to give various similarity degree.

2. DEFINITIONS

Definition 1. Let U be an universe. $F(U) = \{A | A$ is fuzzy set on $U\}$. The fuzzy set $e \in F(F(U))$ is called a evaluator on $F(U)$ if it satisfies the following conditions:

(1) $e(\phi) = 0$, $e(U) = 1$,

(2) if $A \subseteq B \in F(U)$ then $e(A) \leq e(B)$.

Definition 2. $d \in F(F(U) \times F(U))$ is called a fuzzy distance on $F(U)$ (shall is called diatance), if d satisfies the following:

1) $d(A,A) = 0$, $d(\phi,U) = 1$,

2) $d(A,B) = d(B,A)$,

3) from $A \leq B \leq C$ follows that

$$d(A,B) \vee d(B,C) \leq d(A,C).$$

Definition 3. $S \in F(F(U) \times F(U))$ is called a similarity degree on $F(U)$, if it is defined as

1) $S(A,A) = 1$, $S(\phi, U) = o$,

2) $S(A,B) = S(B,A)$.

3) from $A \leq B \leq C$ follows that

$$d(A,B) \wedge d(B,C) \geq d(A,C).$$

3. THE RELATIONS AMONG EVALUATOR, DISTANCE AND SIMILARITY DEGREE AND THEOREMS ABOUT THEIR GENERATION

Theorem 1. The relation between similarity degree and distance is mutual remainder.

Theorem 2. If $S(A,B)$ ($d(A,B)$) is a smilarity degree (distance) on $F(U)$, then $S(U,A)$ and $S(\phi, \bar{A})$ ($d(\phi,A)$ and $d(U,\bar{A})$) are evaluators on $F(U)$.

Where $A(U)$ is used to express grade of membership of U in A, $\bar{A}$ is also a fuzzy set on U, as well as $\bar{A}(u)=1-A(u)$.

Let U,V be two universes; suppose $T = \{ f_u \mid u \in U, f_u \in F(F(V)) \}$ then T determines a mapping f_T from $F(V)$ to $F(U)$, when we define as following,

$$f_T(A)(u) = f_u(A).$$

Theorem 3. If e and f_u are evaluators on $F(U)$ and $F(V)$, respectively, then $e \circ f_T$ is a evaluator on $F(V)$.

Here the "$\circ$" is the composition of functions.

Proof. Because f_u is an evaluator, then $f_u(\phi) = 0$, $f_T(\phi)(u) = f_u(\phi) = 0$ for $\forall u \in U$ holds. Such $f_T(\phi) = \phi$; therefore $(e \circ f_T)(\phi) = e(f_T(\phi)) = e(\phi) = 0$, From $f_T(V)(u) = f_u(V) = 1$, we have $f_T(V) = U$, so that $(e \circ f_T)(V) = e(U) = 1$. On the other hand, for $A \leq B \in F(V)$ holds

$$f_T(A)(u) = f_u(A) \leq f_u(B) = f_T(B)(u),$$

i.e. $f_T(A) \leq f_T(B)$ by definition of e we have

$$(e \circ f_T)(A) \leq (e \circ f_T)(B).$$

The proof of the rest part is easily obtained.

Let $T = \{ g_u \mid u \in U,\ g_u \in F(F(V) \times F(V)) \}$ then T determines a mapping g_T from $F(V) \times F(V)$ to $F(U)$, when we define

$$g_T(A,B)(u) = g_u(A,B).$$

Theorem 4. If e is an evaluator on $F(U)$, g_u is similarity degree (distance), then $e \circ g_T$ is similarity degree (distance) on $F(V)$.

Proof. We only prove in part. Let $A \leq B \leq C \in F(V)$, because g_u is a similarity degree on $F(V)$, therefore

$$g_u(A,C) \leq g_u(A,B) \wedge g_u(B,C),$$

such that for $\forall u \in U$ holds

$$g_T(A,C)(u) = g_u(A,C) \leq g_u(A,B) = g_T(A,B)(u),$$

i.e. $g_T(A,C) \leq g_T(A,B)$. Since e is evaluator we have $(e \circ g_T)(A,C) \leq (e \circ g_T)(A,B)$ similarity we can write

$$(e\circ g_T)(A,C) \le (e\circ g_T)(B,C) ,$$

from above it follows that

$$(e\circ g_T)(A,C) \le (e\circ g_T)(A,B) \wedge (e\circ g_T)(B,C) .$$

Remark. When U , V are finite universes, the above theorem can be expressed as:

If $e(x_1,\cdots,x_n)$ is evaluator, $g_i(y_1,\cdots,y_m)$ are similarity degrees (distances) , $i=1,\cdots,n$; then $e(g_1,\cdots,g_n)$ is a similarity degree (distance).

As above it is easy to see that sometimes if similarity degree does not concerns with specific universes it is even more convenient. For the sake of convenience we can call m-place similarity degree, n-place distance etc.

Theorem 5. Let $|A - B|$ be a fuzzy set on U such

$$|A-B|(u) = A(u) \vee B(u) - A(u) \wedge B(u) ,$$

then $e(|A-B|)$ is a distance on $F(U)$.

Following we let $F_R(U)$ be the set of all real value function on U .

Theorem 6. Let $f(X) \in F_R(U)$ be monotone increasing as well as for the certain A, B holds $f(A) \neq f(B)$, then

$$e(X) = \frac{f(A+(B-A)X) - f(A)}{f(B) - f(A)}$$

is a evaluator on $F(U)$.

where AB, A+B are defined as

$$(AB)(u) = A(u)\ B(u)$$

$$(A+B)(u) = A(u) + B(u)$$

respectively.

4. EXAMPLES

Since above theorems we can see that the evaluator and similarity degree can be very easily constructed. For the sake of the generating of general similarity degree, we first give the best simple evaluator as follows

1. x^{μ} , $(\mu > 0)$;

2. $\dfrac{a^{x}-1}{a-1}$, $(a > 1)$;

3. $\log_a((a-1)x+1)$, $(a > 1)$;

4. $\dfrac{\cos(\frac{\beta-\alpha}{2}x+\alpha)\sin\frac{\beta-\alpha}{2}x}{\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}}$, $(-\frac{\pi}{2} \leq \alpha < \beta \leq \frac{\pi}{2})$;

5. $\frac{1}{\beta}\arcsin(x\sin\beta)$, $(0 < \beta \leq \frac{\pi}{2})$;

6. $\dfrac{tg\,\beta x}{tg\,\beta}$, $(0 < \beta < \frac{\pi}{2})$;

7. $\frac{1}{\beta}\operatorname{arctg}(x\,tg\,\beta)$, $(0 < \beta < \frac{\pi}{2})$.

Let f($x_1, \dots, x_n$) be a n-place real function whose constant term equals zero; it is expressed

$$f(X) = \sum_{\beta_1,\dots,\beta_n} \alpha_{\beta_1,\dots,\beta_n} x_1^{\beta_1}\cdots x_n^{\beta_n} .$$

If $\beta_1,\dots,\beta_n > 0$, $\alpha_{\beta_1,\dots,\beta_n} \geqslant 0$, $\sum_{\beta_1,\dots,\beta_n} \alpha_{\beta_1,\dots,\beta_n} = 1$, then $f(X)$ is an evaluator. Specially, $f_1(X) = \sum_{i=1}^{n} \alpha_i x^{i}$ and $f_2(X) = x_1 x_2 \cdots x_n$ are evaluators. If $e_1,\dots,e_n$ are evaluators, so is $f_i(e_1,\dots,e_n)$. These present the following important facts.

1) If weight number is greater than zero as well as it's sum is equal to 1, then weight sum of evaluators is also evaluator.

2) Product of multiplication of evaluators is an evaluator.

Thus $x_i^{\beta_i}$, $(\beta_i > 0)$, $\sum_{i=1}^{n} \alpha_i x_i^{\beta_i}$, $(\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1)$. and $(\sum_{i=1}^{n} \alpha_i x_i^{\beta_i})^{r}$, $(r > 0)$ are evaluators. By above theorems we can obtain that $d(A,B) = (\sum_{i=1}^{n} \alpha_i |a_i - b_i|^{\beta_i})^{r}$ is distance. Where $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)$ are finite fuzzy sets, $r > 0$.

The $d(A,B)$ is extension of well-known classical distance. The α_i, β_i represent that the component of fuzzy vector has an effect on similarity degree no equality weight. Therefore it must be a better mathematical model.

By above results we give different various evluators very arbitrarilly e.g.

$$f(X) = \sum_{i=1}^{n} \alpha_i \left(\sin \frac{\pi x_i}{2}\right)^{\beta_i},$$

$$g(X) = \prod \sin \frac{\pi x_i^{\beta_i}}{2}.$$

$$(\alpha_i \geq 0, \quad \beta_i > 0 \quad \sum_{i=1}^{n} \alpha_i = 1)$$

The similarity degree got useful above is not seen in historical document.

Remark. The similarity degree given in our paper must be different from known on computation, but today the computer is broadly applied; thus in computation they are not intrinsieally different. But the last must be better applied in future.

[1] Wang Peizhuang. Theory and Applications of Fuzzy Set. Printed in China, 1983.