A DECISION MODEL UNDER GENERAL INFORMATION

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1. INTRODUCTION.-

Concepts of lower and upper probability and mathematical expectation was given by Dempster in [2]. Such concepts can be also considered into the field of Shafer's Theory of Evidence. In this way, important contributions can be seen in [3], [4], [5] and [7].

As is known, when on a (finite) referential set X we consider an application $h:X \longrightarrow R^+$ and a basic probability assignment (bpa) $m:\mathcal{P}(X) \longrightarrow [0,1]$, it can be obtain the so called lower and upper integrals of h with respect to m, in a parallel way to that of Dempster.

This paper is devoted to study some properties of such lower and upper integrals in several ways.

Section 2 introduces some basic concepts. Then it is considered the situation in which we have two informations about some property on X, both them represented by bpa's. For such case a relation of inclusion between such bpa's is proposed. Some relations between their associated dual measures (belief and plaussibility ones) are given.

In Section 3 it is considered the case in which we have two, or more, applications $h:X \longrightarrow R^+$ and one bpa $m:\mathcal{P}(X) \longrightarrow [0,1]$. Relations between the respectives lower and upper integrals of sum, inf, sup of h's with respect to m are showed. Finally, when two included bpa's (in sense of Section 2) are considered, some properties of the correspon-ing lower and upper integrals are given.

2. FUZZY MEASURES.-

This section is devoted to give the basic definitions about fuzzy measures and the necessary concepts from Theory of Evidence. In the remainder X will be a finite referential set and $\mathcal{P}(X)$ will denote the set of subsets of X.

Definition 1.- A fuzzy measure on X is an application,

$$g:\mathcal{P}(X) \longrightarrow [0,1]$$

such that,

a) $g(\emptyset) = 0$, $g(X) = 1$

b) $A,B \in \mathcal{P}(X)$ if $A \subset B \Rightarrow g(A) \leq g(B)$

Definition 2.- Given a fuzzy measure g, g^* is said its dual measure if it verifies,

$$\forall A \in \mathcal{P}(X) \Rightarrow g^*(A) = 1 - g(\bar{A})$$

Remark 1: As is known, from the Shafer's point of view, any information about an unknown $x \in X$ can be represented by means of a bpa. In this way, from the lot of measures verifying the above conditions, we shall only consider those related with Shafer's Theory of Evidence.

Definition 3.- A Basic Probability Assignment (BPA) is an application,

$$m:\mathcal{P}(X) \longrightarrow [0,1]$$

verifying,

a) $m(\emptyset) = 0$

b) $\sum_{A \subset X} m(A) = 1$

Thus, with respect to some bpa m two dual measures associated to it can be defined,

Definition 4.- Let m be a bpa on X, then

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \ , \ Bel(A) = \sum_{B \subset A} m(B) \ , \ \forall A \in \mathcal{P}(X)$$

defines the Plaussibility and Belief measures associated to m, respectively.

Both fuzzy measures, from a same bpa m, are dual measures ([3]), that is

$$Pl(A) = 1 - Bel(\bar{A}), \quad \forall A \in \mathcal{P}(X)$$

Consider now two informations on X, each of them represented by one bpa (m_1 and m_2). In [5] was studied the case in which one of those informations is contained in the another one (which we shall denote $m_1 \subset m_2$). This relation of inclusion must be understood in the sense of the knowledge provided by m_1 is less precise than given by m_2, that is

Definition 5.- Given two bpa's m_1 and m_2 we say the evidence represented by m_1 is included in that one by m_2, if

$$\forall A \subset X \quad \exists m_A \ / \ m_A : \mathcal{P}(X) \longrightarrow [0,1]$$

verifying

$$m_1(A) = \sum_B m_A(B) \quad , \forall B \subset A$$

$$m_2(B) = \sum_A m_A(B) \quad , \forall A \supset B$$

Remark 2: This definition is based in the intuitive idea of one additional information (being compatible with the previous one) about an unknown $x \in X$ must to produce an atomization of the evidence.

Thus, in accordance with Definition 5, we can give the following remarkable properties,

Property 1.- Consider two bps's m_1 and m_2 such that $m_1 \subset m_2$. Let Pl_1 and Pl_2 be its plaussibility measures associated. Then for each $C \subset X$, $Pl_1(C) \geq Pl_2(C)$.

Proof:

$$\forall C \subset X \quad Pl_2(c) = \sum_{C \cap B \neq \emptyset} m_2(B) = \sum_{C \cap B \neq \emptyset} \left(\sum_{A \supset B} m_a(B) \right)$$

But,

$$A \subset X,\ B \subset A,\ C \cap B \neq \emptyset \Rightarrow C \cap A \neq \emptyset$$

Hence,

$$\sum_{C \cap B \neq \emptyset} \left(\sum_{B \subset A \subset X} m_A(B) \right) \leq \sum_{C \cap A \neq \emptyset} \left(\sum_{B \subset A} m_A(B) \right) = \sum_{C \cap A \neq \emptyset} m_1(A) = Pl_1(C)$$

Property 2.- Consider two bpa m_1 and m_2 such that $m_1 \subset m_2$. Let Bel_1 and Bel_2 be its Belief measures associated. Then for each $C \subset X$, $Bel_1(C) \leq Bel_2(C)$.

The proof is immediate from Property 1 taking into account that $Bel_i(C) = 1 - Pl_i(C)$, $i = 1,2$.

3. UPPER AND LOWER EXPECTED VALUES.-

In 1967 Dempster generalized the concepts of probability and mathematical expectation by means of the definitions of upper and lower probability and mathematical expectation. Later these operators have been considered in the fields of fuzzy measures and Theory of Evidence ([5], [4], [7],...). In the following we shall give some properties related to the Dempster's operators when a bpa on X is considered.

Definition 6.- Given an application $h : X \longrightarrow R^+$ and a bpa m on X, the upper and lower integrals of h with respect to m are respectively defined by,

$$I^{*}(h/m) = \sum_{A\subset X} m(A) \sup_{a\in A} h(a)$$

$$I_{*}(h/m) = \sum_{A\subset X} m(A) \inf_{a\in A} h(a)$$

Remark 3: It is easy to check that when m is of probabilistic kind, that is $m(\{a\}) = p(a)$ $\forall a\in X$, then

$$I_{*}(h/m) = I^{*}(h/m) = \int_X h\, dp$$

showing as the above integrals constitutes a true generalization of the mathematical expectation concept.

The next Propositions shows some interesting properties about $I_{*}(\cdot)$ and $I^{*}(\cdot)$. We shall only prove them for $I^{*}(\cdot)$, being analogous the proofs for $I_{*}(\cdot)$.

Consider two any applications $h_i : X \to R^{+}$, i = 1,2, and a bpa m on X, $m:\mathcal{P}(X) \to [0,1]$. Then the following Propositions holds,

<u>Proposition 1</u>.-

$$\text{a) } I^{*}(h_1+h_2/m) \leqslant I^{*}(h_1/m) + I^{*}(h_2/m)$$

$$\text{b) } I_{*}(h_1+h_2/m) \geqslant I_{*}(h_1/m) + I_{*}(h_2/m)$$

Proof:

$$I^{*}(h_1+h_2/m) = \sum_{A\subset X} m(A) \sup_{a\in A}\{(h_1+h_2)(a)\} \leqslant \sum_{A\subset X} m(A) \sup_{a\in A} h_1(a) + \sum_{A\subset X} m(A) \sup_{a\in A} h_2(a) =$$

$$= I^{*}(h_1/m) + I^{*}(h_2/m)$$

<u>Proposition 2</u>.-

$$I^{*}(bh_1/m) = \begin{cases} bI^{*}(h_1/m) & \text{if } b > 0 \\ 0 & \text{if } b = 0 \\ |b| I_{*}(h_1/m) & \text{if } b < 0 \end{cases}$$

$$I_{*}(bh_1/m) = \begin{cases} bI_{*}(h_1/m) & \text{if } b > 0 \\ 0 & \text{if } b = 0 \\ |b| I^{*}(h_1/m) & \text{if } b < 0 \end{cases}$$

Proof. Being trivial the case $b \geqslant 0$, we shall prove the Proposition when $b < 0$. Then,

$$I^{*}(bh_1/m) = \sum_{A\subset X} m(A) \sup_{a\in A}(bh_1(a))$$

But,

$$\operatorname*{Sup}_{\substack{a \in A \\ b < 0}} [b h_1(a)] = |b| \operatorname*{Inf}_{a \in A} h_1(a)$$

Hence,

$$I^*(bh_1/m) = |b| \sum_{A \subset X} m(A) \operatorname*{Inf}_{a \in A} h_1(a) = |b| I_*(h_1/m)$$

Proposition 3.- If for any $a \in X$, $h_1(a) \leq h_2(a)$ then

$$I^*(h_1/m) \leq I^*(h_2/m)$$

$$I_*(h_1/m) \leq I_*(h_2/m)$$

Proof: Being $h_1(a) \leq h_2(a), \forall a \in X$, then

$$\operatorname{Sup} h_1(a) \leq \operatorname{Sup} h_2(a) \; ; \; \operatorname{Inf} h_1(a) \leq \operatorname{Inf} h_2(a)$$

and as by definition $m(A) \geq 0, \forall A \subset X$, the proof is follows.

Proposition 4.-

$$I^*(h_1 \vee h_2/m) \geq \operatorname{Max}\{I^*(h_1/m), I^*(h_2/m)\}$$

$$I_*(h_1 \vee h_2/m) \geq \operatorname{Max}\{I_*(h_1/m), I_*(h_2/m)\}$$

Proof:

$$I^*(h_1 \vee h_2/m) = \sum_{A \subset X} m(A) \operatorname*{Sup}_{a \in A} \{\operatorname{Max}[h_1(a), h_2(a)]\} =$$

$$= \sum_{A \subset X} \operatorname{Max} m(A) [\operatorname*{Sup}_{a \in A} h_1(a), \operatorname*{Sup}_{a \in A} h_2(a)] \geq$$

$$\geq \operatorname{Max}[\sum_{A \subset X} m(A) \operatorname*{Sup}_{a \in A} h_1(a) , \sum_{A \subset X} m(A) \operatorname*{Sup}_{a \in A} h_2(a)] =$$

$$= \operatorname{Max}[I^*(h_1/m), I^*(h_2/m)]$$

Proposition 5.-

$$I^*(h_1 \wedge h_2/m) \leq \operatorname{Min}[I^*(h_1/m), I^*(h_2/m)]$$

$$I_*(h_1 \wedge h_2/m) \leq \operatorname{Min}[I_*(h_1/m), I_*(h_2/m)]$$

The proof is as in Proposition 4.

Proposition 6.-

$$I^*(h_1 \vee h_2/m) \geq I^*(h_1 \wedge h_2/m) \geq I_*(h_1 \wedge h_2/m)$$

$$I^*(h_1 \vee h_2/m) \geq I_*(h_1 \vee h_2/m) \geq I_*(h_1 \wedge h_2/m)$$

The proof is folloed taking into account the classical properties of the operators Sup and Inf and that $I^*(h/m) \geq I_*(h/m)$ for any application $h(\cdot)$.

Proposition 7.-

$$I^*(h_1 \wedge h_2/m) + I^*(h_1 \vee h_2/m) \leq I^*(h_1/m) + I^*(h_2/m)$$

$$I_*(h_1 \wedge h_2/m) + I_*(h_1 \vee h_2/m) \geq I_*(h_1/m) + I_*(h_2/m)$$

Proof: By definition,

$$I^*(h_1 \wedge h_2/m) + I^*(h_1 \vee h_2/m) = \sum_{A \subset X} m(A) \{\sup_{a \in A} (h_1 \wedge h_2)(a)\} + \sum_{A \subset X} m(A) \sup_{a \in A} \{(h_1 \vee h_2)(a)\} \leq$$

$$\leq \sum_{A \subset X} m(A) \operatorname{Inf} [\sup_{a \in A} h_1(a), \sup_{a \in A} h_2(a)] + \sum_{A \subset X} m(A) \sup [\sup_{a \in A} h_1(a), \sup_{a \in A} h_2(a)]$$

But as,

$$\operatorname{Inf} [\sup_{a \in A} h_1(a), \sup_{a \in A} h_2(a)] + \sup [\sup_{a \in A} h_1(a), \sup_{a \in A} h_2(a)] = \sup_{a \in A} h_1(a) + \sup_{a \in A} h_2(a)$$

and, moreover, $m(A) \geq 0$, $A \subset X$, the proof is folloed.

Proposition 8.- Let h be an application $h: X \longrightarrow R^+$ and $m_i : \mathcal{P}(X) \longrightarrow [0,1]$, i = 1,2, two bpa's such that $m_1 \subset m_2$. Then,

$$I^*(h/m_1) \geq I^*(h/m_2)$$

$$I_*(h/m_1) \leq I_*(h/m_2)$$

Proof: Since definition of $I^*(\cdot)$ and definition 5,

$$I^*(h/m_1) = \sum_{A \subset X} m_1(A) \sup_{a \in A} h(a) = \sum_{A \subset X} (\sum_{A \subset B} m_A(B)) \sup_{a \in A} h(a) =$$

$$= \sum_{B \subset X} \sum_{A \supset B} m_A(B) \sup_{a \in A} h(a) \geq \sum_{B \subset X} \sum_{A \supset B} m_A(B) \sup_{a \in B} h(a) =$$

$$= \sum_{B \subset X} m_2(B) \sup_{a \in B} h(a) = I^*(h/m_2)$$

Proposition 9.- Let h be an application $h: X \longrightarrow R^+$ and $m_i : \mathcal{P}(X) \longrightarrow [0,1]$, i = 1,2, two bpa's. Then for any $\alpha \in [0,1]$,

$$I^*(h/\alpha m_1 + (1-\alpha) m_2) = \alpha I^*(h/m_1) + (1-\alpha) I^*(h/m_2)$$

$$I_*(h/\alpha m_1 + (1-\alpha) m_2) = \alpha I_*(h/m_1) + (1-\alpha) I_*(h/m_2)$$

Proof:

$$I^*(h/\alpha m_1 + (1-\alpha) m_2) = \sum_{A \subset X} [\alpha m_1(A) + (1-\alpha) m_2(A)] \sup_{a \in A} h(a) =$$

$$= \sum_{A \subset X} [\alpha m_1(A) \sup_{a \in A} h(a) + (1-\alpha) m_2(A) \sup_{a \in A} h(a)] =$$

$$= \alpha I^*(h/m_1) + (1-\alpha) I^*(h/m_2)$$

Proposition 10.- For any application $h:X \longrightarrow [0,1]$ and any bpa $m:\mathcal{P}(X) \longrightarrow [0,1]$, it is verified,

$$I^{*}(h/m) + I_{*}(1-h/m) = 1$$

Proof:

$$I_{*}(1-h/m) = \sum_{A\subset X} m(A) \operatorname{Inf}_{a\in A}(1-h(a)) = \sum_{A\subset X} m(A) [1-\operatorname{Sup}_{a\in A} h(a)] =$$

$$= \sum_{A\subset X} m(A) - \sum_{A\subset X} m(A) \operatorname{Sup}_{a\in A} h(a) = 1 - I^{*}(h/m)$$

FINAL COMMENTS.-

If we denote m_o and m_p the bpa's corresponding to the total ignorance and to the probabilistic kind, respectively, then for any another bpa m, $m_o \subset m \subset m_p$. Thus, if we have two bpa's m_1 and m_2, such that $m_1 \subset m_2$, it is evident that $m_o \subset m_1 \subset m_2 \subset m_p$.

Moreover,

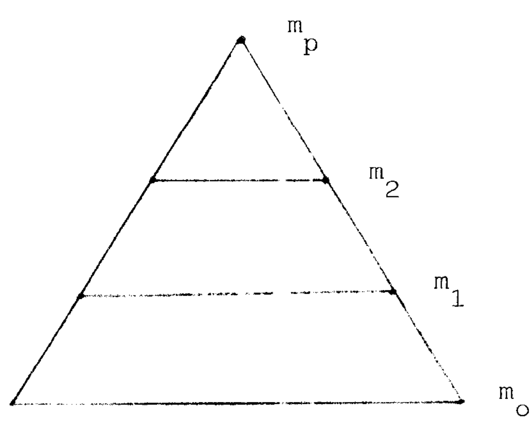
$$m_1 \subset m_2 \Rightarrow I^{*}(h/m_1) \geq I^{*}(h/m_2) \ ; \ I_{*}(h/m_1) \leq I_{*}(h/m_2)$$

for any $h:X \longrightarrow R^{+}$. Then, the inclusion of intervals,

$$[I_{*}(h/m_2), I^{*}(h/m_2)] \subset [I_{*}(h/m_1), I^{*}(h/m_1)]$$

it is verified.

This situation can be described by the following scheme,



where m_p is reducted to one point because its probabilistic kind.

From this context several applications can be carried out. In this way, an approach to Decision Making problems by bpa's will be the matter of a forthcoming paper.

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