COMPARISON OF FLAT FUZZY NUMBERS

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ABSTRACT : The paper deals with the problem of comparing n flat fuzzy numbers representing n intervals whose bounderies are not sharp.

The comparison is obtained using : (i) a crisp interval order representation which introduces a crisp transitive preference and a crisp intransitive indifference among the fuzzy numbers, (ii) two linear orders at minimum symmetric distance from the crisp interval order.

All these preference structures depend on a threshold related to the degree of possibility of a flat fuzzy number being greater or equal to another flat fuzzy number.

The solution is compared to the four grades of dominance introduced by DUBOIS and PRADE using an example. One of these grades is shown to be a fuzzy interval order (complete and Ferrers fuzzy relation).

I. INTRODUCTION

Many authors have investigated ordering fuzzy numbers (see ADAMO /1/, BAAS and KWAKERNAAK /2/, BALDWIN and GUILD /3/, BUCKLEY /5/, CHANG /6/, DUBOIS and PRADE /7/, /8/, EFTATHIOU and TONG /9/, JAIN /12/, KERRE /13/, SHIMURA /17/, YAGER /18/ and two reviews by BORTOLAN and DEGANI /4/ and by FREELING /11/).

Most of these authors suggest to define a ranking function, mapping each fuzzy set (corresponding to each fuzzy number) into the real line. Let us suppose that we have a family A of fuzzy numbers $M_i = \{x,\mu_i(x)\}$, $i\varepsilon\{1,2,3,\ldots,n\}$, $x\varepsilon R$, where $\mu_i(x)$ represents the degree of membership of x in M_i. If F is a real-valued mapping from the set of fuzzy subsets to R, one obtains easily a quasi-order structure $(>,\approx)$ on the set $\{M_i\}$ using

$$M_i > M_j \text{ iff } F(M_i) > F(M_j)$$

$$M_i \approx M_j \text{ iff } F(M_i) = F(M_j)$$

In this paper, we consider the Θ-level sets related to $\{M_i\}$ which are the sets $I(M_i,\Theta) = \{x:\mu_i(x)\geq\Theta\}$ and we investigate the family of preference structures $\{\overset{\theta}{>},\overset{\theta}{\approx};\ 0\leq\Theta\leq 1\}$ such that

$$M_i \overset{\theta}{>} M_j \text{ iff } I(M_i,\Theta) > I(M_j,\Theta),$$

$$M_i \overset{\theta}{\approx} M_j \text{ iff } I(M_i,\Theta) \cap I(M_j,\Theta) \neq \phi.$$

where $I(M_i,\Theta) > I(M_j,\Theta)$ iff $x>y$, $\forall\, x\varepsilon I(M_i,\Theta)$, $\forall\, y\varepsilon I(M_j,\Theta)$.

The parameter Θ allows us to identify some crisp indifference threshold. When Θ is decreasing from 1 to 0 the range of indifference is increasing. Using the intervals $I(M_i,\Theta)$, the comparison of flat fuzzy numbers is then transformed into the classical problem of crisp interval classification.

2. Θ-INTERVALS RELATED TO FLAT FUZZY NUMBERS

Let us consider n L-R flat fuzzy numbers (see DUBOIS and PRADE /8/) briefly denoted

$$M_i = (m_{1i}, m_{2i}, \alpha_i, \beta_i)_{LR} \text{ where}$$

$$M_i = \{m, \mu_i(x)\} \text{ and } \mu_i(x) = L((m_{1i}-x)/\alpha_i),\ x \leq m_{1i},\ \alpha_i > 0,$$
$$= R((x-m_{2i})/\beta_i),\ x \geq m_{2i},\ \beta_i > 0,$$
$$= 1 \text{ otherwise.}$$

L and R are reference functions respectively non decreasing on $(-\infty, m_{1i}]$ and non increasing on $[m_{2i}, \infty)$.

If $0 \leq \Theta \leq 1$, let us consider the Θ-level sets $I(M_i,\Theta)$ related to the flat fuzzy numbers M_i. Due to the monotonic structure of L and R, these Θ-level sets are open intervals of the real line with origin $g_i(\Theta)$ and extremity $f_i(\Theta)$ and

$$I(M_i,\Theta_1) \supseteq I(M_i,\Theta_2) \supseteq (m_{1i}, m_{2i}) = I(M_i,1) \text{ for all } \Theta_1 \geq \Theta_2$$

Following DUBOIS and PRADE /8/ we consider the degree of possibility (also called "grade of possibility of dominance") of $M_i \geq M_j$ which is defined as

$$PD(M_i,M_j) = \text{Poss.}(M_i \geq M_j) = \sup_{x,y:x \geq y} \min(\mu_i(x), \mu_j(y))$$

It can be easily seen that (see Fig.1) :

$$\text{Poss.}(M_i \geq M_j)=1 \text{ and Poss.}(M_j \geq M_i) = \text{hgt}(M_i \cap M_j) \text{ iff } m_{2j} \leq m_{1i}$$

Figure 1

Moreover,

$\text{Poss.}(M_i \geq M_j)=1$ and $\text{Poss.}(M_j \geq M_i) < \Theta$ iff

$I(M_i,\Theta) > I(M_j,\Theta)$

or, in an equivalent way, iff

$g_i(\Theta) > f_j(\Theta)$

It derives from these results that

$M_i \overset{\theta}{>} M_j$ iff Poss.$(M_i \geq M_j)=1$ and Poss.$(M_j \geq M_i)<\theta$,

$M_j \overset{\theta}{>} M_i$ iff Poss.$(M_j \geq M_i)=1$ and Poss.$(M_i \geq M_j)<\theta$,

$M_i \overset{\theta}{\approx} M_j$ otherwise.

It turns out that these rules correspond to the procedure proposed in /5/ and /8/.

If we consider $F_\theta(M_i) = f_i(\theta)$ we obtain the ranking rules proposed by ADAMO /1/. The advantage of the method proposed in this section comes from the fact that preferences are based on right and left parts of memberships.

Recalling the fundamental representation theorem of FISHBURN /14/, the family of preference relations $\{(\overset{\theta}{>},\overset{\theta}{\approx})\}$ is a family of interval orders. In the next section we present some results related to such families.

3. SOME PROPERTIES RELATED TO THE FAMILIES OF INTERVAL ORDERS

Let us consider a matrix $\mathcal{M}^\theta$ with elements $\{M^\theta_{ij}\}$, $\forall\, i,j \in \{1,\ldots,n\}$ where $M^\theta_{ij}=1$ iff $M_i \overset{\theta}{>} M_j$ or $M_i \overset{\theta}{\approx} M_j$,

$M^\theta_{ij}=0$ otherwise.

If S^θ represents a crisp binary relation such that

$M_i\ S^\theta\ M_j$ iff $M^\theta_{ij}=1$,

not $(M_i\ S^\theta\ M_j)$ iff $M^\theta_{ij}=0$,

S^θ presents a total interval order structure and the preference structure $\{\overset{\theta}{>},\overset{\theta}{\approx}\}$ associated to S^θ can be interpreted as

$$M_i \overset{\theta}{>} M_j \quad \text{iff} \quad M_i\ S^\theta\ M_j \text{ and not } (M_j\ S^\theta\ M_i)$$

$$M_i \overset{\theta}{\approx} M_j \quad \text{iff} \quad M_i\ S^\theta\ M_j \text{ and } M_j\ S^\theta\ M_i$$

$\overset{\theta}{>}$ is an asymmetric relation called "strict preference" and $\overset{\theta}{\approx}$ is a reflexive and symmetric indifference relation. Looking only at the strict preference, one can define a matrix $\mathcal{P}^\theta$ with elements

$$\{P^\theta_{ij}\},\ \forall\, i,j\varepsilon\{1,\ldots,n\} \text{ where } P^\theta_{ij}=1 \quad \text{iff} \quad M_i \overset{\theta}{>} M_j$$
$$=0 \quad \text{otherwise.}$$

It can be easily seen that the grade of possibility of dominance

$$PD(M_i,M_j) = \text{Poss.}(M_i \geqq M_j) = \max_\theta\ \theta\ M^\theta_{ij}$$

Furthermore, $S^{.5}$, the (α=.5)-level set of PD is an interval order which minimizes the Hamming distance between PD and all possible crisp relation T on A, i.e.

$$S^{.5} \text{ minimizes } \sum_{i,j\varepsilon\{1,\ldots,n\}} |PD(M_i,M_j)-\mu_T(M_i,M_j)|$$

where $\mu_T(M_i,M_j)=1$ if $M_i\ T\ M_j$,

$=0$ otherwise.

The proof is obvious and can be found in /15/.

We now consider the following proposition :

The fuzzy relation PD presents a fuzzy interval order structure, i.e. S^θ being a strongly complete Ferrers relation, for all $\theta\ \varepsilon[0,1]$, implies that PD is a fuzzy complete Ferrers relation.

If we recall that a crisp relation S^θ on the set A is

- *strongly complete* provided that $M_i\ S^\theta\ M_j$ or $M_j\ S^\theta\ M_i$, $\forall\, i,j\varepsilon\{1,\ldots,n\}$
- *Ferrers* provided that $M_i\ S^\theta\ M_j$ and $M_k\ S^\theta\ M_\ell \Rightarrow M_i\ S^\theta\ M_\ell$ or $M_k\ S^\theta\ M_j$, $\forall\, i,j,k,\ell\ \varepsilon\{1,\ldots,n\}$

and that a fuzzy relation PD on the set A is

- *complete* provided that $\max\{PD(M_i,M_j),\ PD(M_j,M_i)\} = 1,\ \forall\ i,j \in \{1,\dots,n\}$
- *Ferrers* provided that $\min\{PD(M_i,M_j),\ PD(M_k,M_\ell)\} \leq \max\{PD(M_i,M_\ell), PD(M_k,M_j)\}$ $\forall\ i,j,k,\ell \in \{1,\dots,n\}$

the assertion is obvious. S^θ is clearly strongly complete and Ferrers (see /14/). Suppose $PD(M_i,M_j)=\Theta_1 \geq PD(M_k,M_\ell)=\Theta_2$. Then

$$M^{\theta_2}(k,\ell) = M^{\theta_2}(i,j) = M^{\theta_1}(i,j) = 1 \quad \text{and}$$

$$M_i\ S^{\theta_2}\ M_j \text{ and } M_k\ S^{\theta_2}\ M_\ell \rightarrow M_i\ S^{\theta_2}\ M_\ell \text{ or } M_k\ S^{\theta_2}\ M_j$$

$$PD(M_i,M_\ell) \geq \Theta_2 \text{ or } PD(M_k,M_j) \geq \Theta_2 \quad \text{and}$$

$$\max\{PD(M_i,M_\ell),\ PD(M_k,M_j)\} \geq \Theta_2 = \min\{PD(M_i,M_j),\ P\ (M_k,M_\ell)\}$$

The proof of the completeness is trivial.

It has been shown that the matrix $\mathcal{P}^\theta$ could be, for some linear ordering of the row elements $O^{R,\theta}$ ans some linear ordering of the column elements $O^{C,\theta}$, be presented in an upper-diagonal step-type form (see /14/) like in Fig.2.

Figure 2

These two orderings correspond to

$M_i\ O^{C,\theta}\ M_j$ iff $s_C^\theta(i) < s_C^\theta(j)$ or $s_C^\theta(i) = s_C^\theta(j)$ and $s_R^\theta(i) > s_R^\theta(j)$,

$M_i\ O^{R,\theta}\ M_j$ iff $s_R^\theta(i) > s_R^\theta(j)$ or $s_R^\theta(i) = s_R^\theta(j)$ and $s_C^{(\theta)}(i) < s_C^{(\theta)}(j)$,

where the *scores* $s_C^\theta(i)$ and $s_R^\theta(i)$ are defined as follows :

$$s_C^\theta(i) = \sum_j P_{ji}^\theta$$

$$s_R^\theta(i) = \sum_j P_{ij}^\theta$$

The elements of A such that $s_C^\theta(i)=s_C^\theta(j)$ and $s_R^\theta(i)=s_R^\theta(j)$ are included in a set E^θ with elements being equivalence classes.

It was also shown /16/ that the orderings $O^{R,\theta}$ and $O^{C,\theta}$ for the elements A/E^θ are at minimum symmetric difference distance (crisp Hemming distance) from th interval orders. This important property allows us to consider these rankings close to $(\overset{\theta}{<},\overset{\theta}{\approx})$ as a second possible answer to the problem of construction of the ordering relation on the set of fuzzy members.

At last, if the relation $\overset{\theta}{\approx}$ is analysed as an interval graph (J^θ,I^θ), each interval $I(M_i,\theta)$ being a node of J^θ and connecting two nodes by an edge iff the corresponding intervals intersect, the complement of this undirected graph is called a comparability graph $(J^\theta,\overline{I}^\theta)$. To each edges of $\overline{I}^\theta$ can be assigned a one-way direction given by $\overset{\theta}{>}$ in such a way that the resulting digraph is transitive. Using the FULKERSON and GROSS theorem /10/ the maximal cliques of the interval graph present a ranking such that, for every node a of the comparability graph, the maximal cliques containing a occur consecutively. The ranking of the maximal cliques is a third possible answer to the problem of comparison of fuzzy numbers.

4. *EXAMPLE*

Let us consider the set of flat fuzzy numbers with trapezoidal form $\{M_1,M_2,M_3,M_4,M_5\}$. These numbers are related to a fuzzy ranking using a scale from 0 (lowest) to 10 (highest). If $M_i = \{m_{1i}, m_{2i}, \alpha_i, \beta_i\}$ and L,R correspond to linear tolerances of slope respectively equal to α_i and β_i, we obtain Table 1 and Figure 3.

M_i	m_{1i}	m_{2i}	α_i	β_i
M_1	8.5	9.5	∞	$-\infty$
M_2	6	8	1/3	-1/2
M_3	3	4	1	-1/2
M_4	4.5	5.5	1	-1
M_5	3.5	7	1	-2.3

Table 1

Figure 3

The grades of possibility of dominance PD are given in Table 2.

PD	M_1	M_2	M_3	M_4	M_5
M_1	1	1	1	1	1
M_2	.75	1	1	1	1
M_3	0	.6	1	.833	1
M_4	0	.875	1	1	1
M_5	0	1	1	1	1

Table 2

The matrices $\mathcal{P}^{\theta}$, the sets of equivalence classes and the maximal cliques of (J^{θ}, I^{θ}) related to $\Theta=1$, .75, .5, are represented in Tables 3, 4 and 5 respectively.

$\mathcal{P}^{1}$	M_1	M_2	M_5	M_4	M_3
M_1	0	1	1	1	1
M_2	0	0	0	1	1
M_4	0	0	0	0	1
M_3	0	0	0	0	0
M_5	0	0	0	0	0

Equivalence classes

E^1 : $\{E_1^1=\{M_1\}, E_2^1=\{M_2\}, E_3^1=\{M_3\}, E_4^2=\{M_4\}, E_5^1=\{M_5\}\}$

Maximal cliques

C^1 : $\{C_1^1=\{M_1\}, C_2^1=\{M_2,M_5\}, C_3^1=\{M_4,M_5\},$

$C_4^1=\{M_3,M_5\}\}$

Table 3 ($\Theta=1$)

$\mathcal{P}^{.75}$	M_1	M_2	M_5	M_4	M_3
M_1	0	0	1	1	1
M_2	0	0	0	0	1
M_4	0	0	0	0	0
M_3	0	0	0	0	0
M_5	0	0	0	0	0

Equivalence classes

$E^{.75}$: $\{E^{.75}=\{M_4,M_5\}, E_2^{.75}=\{M_1\}, E_3^{.75}=\{M_2\},$

$E_4^{.75}=\{M_3\}\}$

Maximal cliques

$C^{.75}$: $\{C_1^{.75}=\{M_1,M_2\}, C_2^{.75}=\{E_1^{.75}, M_2\}$

$C_3^{.75}=\{E_1^{.75},M_3\}\}$

Table 4 ($\Theta=.75$)

$\wp^{.5}$	M_1	M_2	M_5	M_4	M_3
M_1	0	0	1	1	1
M_2	0	0	0	0	0
M_4	0	0	0	0	0
M_5	0	0	0	0	0
M_3	0	0	0	0	0

Equivalence classes

$E^{.5}$: $\{E_1^{.5}=\{M_3,M_4,M_5\},\ E_4^{.5}=\{M_1\},\ E_3^{.5}=\{M_2\}\}$

Maximal cliques

$C^{.5}$: $\{C_1^{.5}=\{M_1,M_2\},\ C_2^{.5}=\{M_2,E_1^{.5}\}\}$

Table 5 (Θ=.5)

We finally obtain the following quasi-orderings

$O^{R,1}$: $M_1 > M_2 > M_4 > M_5 > M_3$

$O^{C,1}$: $M_1 > M_2 > M_5 > M_4 > M_3$

$O^{L,.75} \equiv O^{C,.75}$: $M_1 > M_2 > M_5 \approx M_4 > M_3$

$O^{L,.5} \equiv O^{C,.5}$: $M_1 > M_2 > M_5 \approx M_4 \approx M_3$

The "orderings" according to the maximal cliques give :

at the level $\Theta=1$: $\{M_1\} > \{M_2 \approx M_5\} > \{M_4 \approx M_5\} > \{M_3 \approx M_5\}$

$\Theta=.75$: $\{M_1 \approx M_2\} > \{M_2 \approx M_4 \approx M_5\} > \{M_3 \approx M_4 \approx M_5\}$

$\Theta=.5$: $\{M_1 \approx M_2\} > \{M_2 \approx M_3 \approx M_4 \approx M_5\}$

These last results can be interpreted in terms of graph representation (see Figure 4).

Figure 4

5. COMPARISON WITH DUBOIS AND PRADE's GRADES OF DOMINANCE

In /4/, BORTOLAN and DEGAINI conclude that "... the four indices of DUBOIS and PRADE can be conveniently used". Let us consider these indices in relation with the example of section 4.

Using the notations

$$\mu_{[M_i,\infty)}(x) = \sup_{y\leq x} \mu_i(y)$$

$$\mu_{(M_i,\infty)}(x) = \inf_{y\geq x} (1-\mu_i(y))$$

we obtain four grades of dominance (see /7/) :

(1) a grade of possibility of dominance

$$\mu_{PD}(M_i) = \text{Poss.}(M_i \geq \widetilde{\max_{j\neq i}}\, M_j)$$

$$= \sup_x \min\{\mu_i(x), \mu_{[\widetilde{\max}\, M_j,\infty)}(x)\}$$

(2) a grade of possibility of strict dominance

$$\mu_{PSD}(M_i) = \text{Poss.}(M_i > \widetilde{\max_{j\ i}}\, M_j)$$

$$= \sup_x \min\{\mu_i(x),\ \mu_{(\widetilde{\max}\, M_j,\infty)}(x)\}$$

(3) a grade of necessity of dominance

$$\mu_{ND}(M_i) = \text{Nec.}(M_i \geq \widetilde{\max_{j\neq i}}\, M_j)$$

$$= \inf_x \max\{1-\mu_i(x),\ \mu_{[\widetilde{\max}\, M_j,\infty)}(x)\}$$

(4) a grade of necessity of strict dominance

$$\mu_{NSD}(M_i) = \text{Nec.}(M_i > \widetilde{\max_{j\neq i}}\, M_j)$$

$$= \inf_x \max\{1-\mu_i(x),\ \mu_{(\widetilde{\max}\, M_j,\infty)}(x)\}$$

$$= 1-\text{Poss.}(\widetilde{\max_{j\neq i}}\, M_j \geq M_i)$$

$$= 1-\sup_x \min\{\widetilde{\max_{j\neq i}}\, M_j(x), \mu_{[M_i,\infty)}(x)\}$$

For the example given in section 4, we obtain Table 7.

M_i	PD_i	PSD_i	ND_i	NSD_i
M_1	1	.75	1	.25
M_2	.75	.25	0	0
M_3	0	0	0	0
M_4	0	0	0	0
M_5	0	0	0	0

Table 7

It is clear that none of the indices is able to discriminate M_3, M_4, M_5. In fact this situation is due to the "domination" of M_1 which masks M_3, M_4 and M_5.

If M_1 is not considered, we obtain the results of Table 8

M_i	PD_i	PSD_i	ND_i	NSD_i
M_2	1	.857	.625	0
M_3	.6	0	0	0
M_4	.875	0	.375	0
M_5	1	.143	0	0

Table 8

Two conclusions derive from the consideration of the previous results.

(i) The values obtained for the grades of dominance depend strongly on the set of alternatives. If new issues are added to the initial set of alternatives, they effect the final ranking. The ranking in dependant of irrelevant alternatives. It is not the case for the comparison with Θ-level sets : if $M_i \overset{\theta}{>} M_j$ or $M_i \overset{\theta}{\approx} M_j$, M_i and M_j being elements of issues $M_1, \ldots, M_n$, the same conclusion is true for a larger set of alternatives.

(ii) There is however a strong correlation between the ranking proposed in section 4 and those obtained with table 8.

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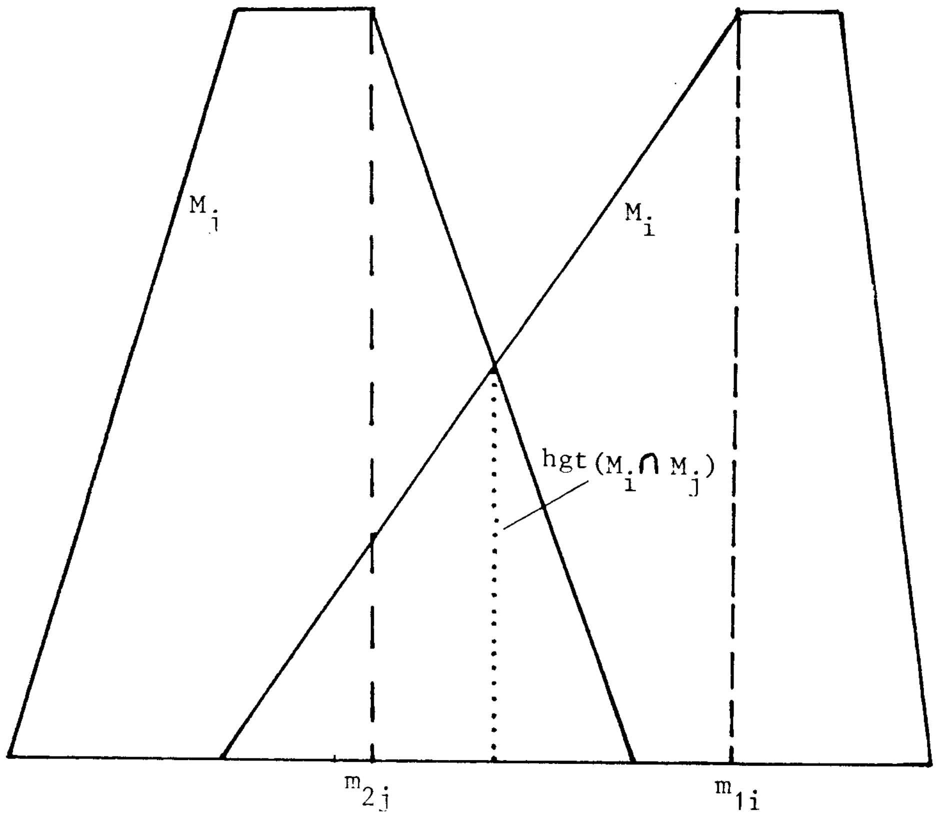

Figure 1

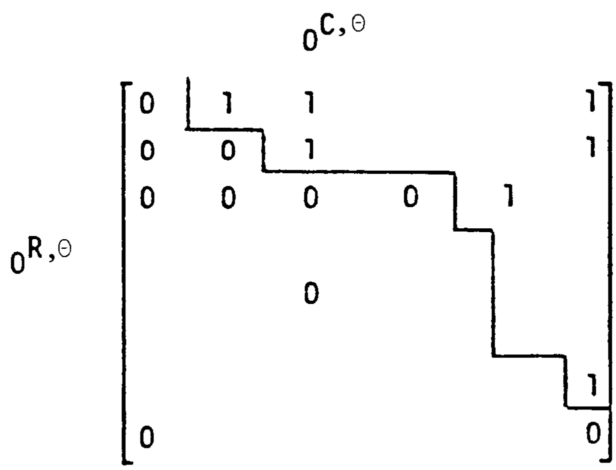

Figure 2

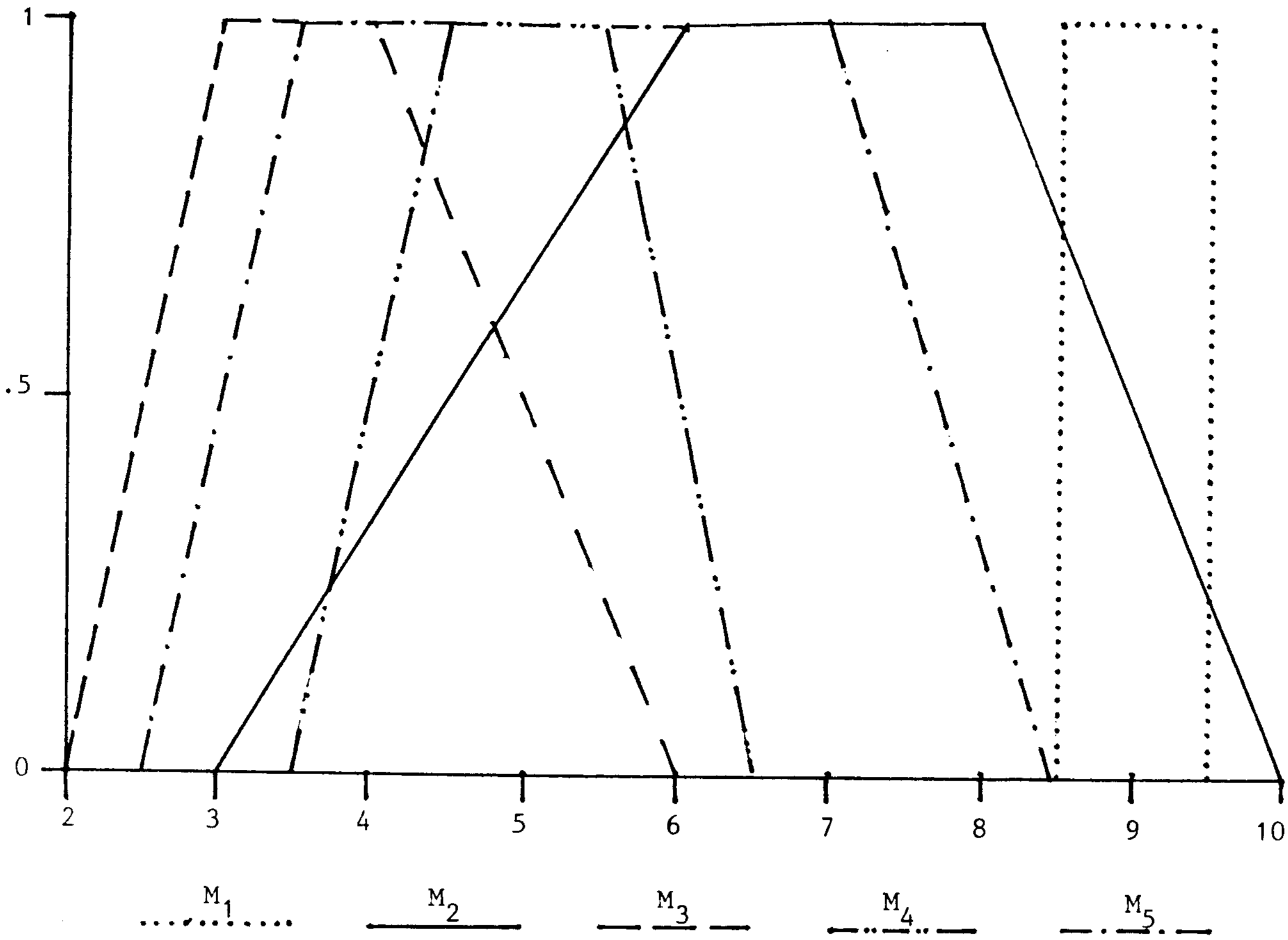

Figure 3

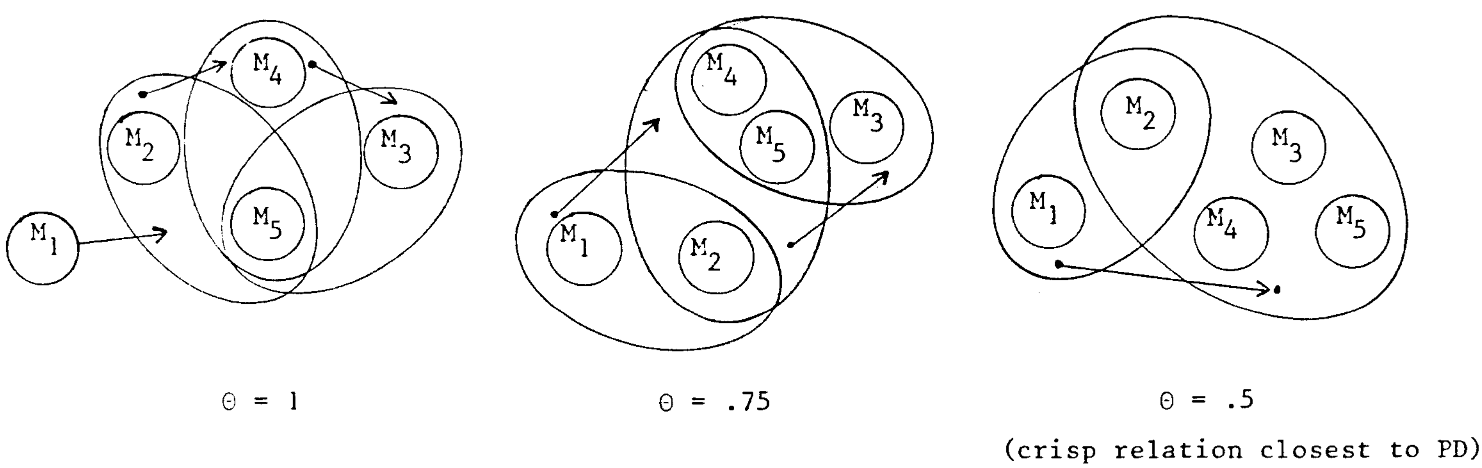

Figure 4