ON ONE RELATIONSHIP BETWEEN CLASSICAL PROBABILITY MEASURE AND FUZZY P-MEASURE

Krzysztof PIASECKI

Department of Mathematics, academy of Ekonomy,
ul. Marchlewskiego 146/150, 60-967 Poznań, Poland.

Let $\sigma = \{\mu : \Omega \to [0,1]\}$ be any **soft** fuzzy σ-algebra i.e. fuzzy σ- algebra (see [1]) uncontaining the fuzzy subset $[\![\frac{1}{2}]\!]_{\Omega}: \Omega \to \{\frac{1}{2}\}$. Since $0_{\Omega} \in \sigma$, the crisp set Ω can be always decomposed as a union

$$\Omega = \Omega_1 \cup \Omega_2 \tag{1}$$

where $\Omega_1 \neq \emptyset$, $\Omega_1 \cap \Omega_2 = \emptyset$ and $\chi_{\Omega_2} \in \sigma$ (the mapping χ_{Ω_2} is a membership function of crisp subset Ω_2) . Obviously, Ω_2 can be empty.

Let Ω_1 be a fixed crisp subset in Ω satisfying (1).

Definition 1: The mapping

$$K(\cdot, \Omega_1) : \sigma \to 2^{\Omega_1} ,$$

defined by the identity

$$\forall \mu \in \sigma \qquad K(\mu, \Omega_1) = \{\omega : \omega \in \Omega_1 , \mu(\omega) > \tfrac{1}{2}\} , \tag{2}$$

is called a support of nonemptiness.

Definition 2: The mapping

$$K^*(\cdot,\Omega_1): \sigma \to 2^{\Omega_1} ,$$

given by the identity

$$\forall \mu \in \sigma \qquad K^*(\mu,\Omega_1) = \{\omega : \omega \in \Omega_1 , \mu(\omega) = \tfrac{1}{2}\} , \qquad (3)$$

is called a support of ill-defined elements.

Let $L(\mu,\Omega_1) = K(\mu,\Omega_1) \cup K^*(\mu,\Omega_1)$ for each $\mu \in \sigma$, $K^*(\sigma,\Omega_1) = \{M: M \in 2^{\Omega_1}, \exists \mu \in \sigma : M = K^*(\mu,\Omega_1)\}$ and $\Omega(\sigma,\Omega_1) = \{M: M \in 2^{\Omega_1}, \exists \mu \in \sigma : M = K(\mu,\Omega_1)\} \cup \{M: M \in 2^{\Omega_1}, \exists \mu \in \sigma , M = L(\mu,\Omega_1)\}$.

Theorem 1: $\Omega(\sigma,\Omega_1)$ is a σ-algebra in Ω_1. Furthermore, $K^*(\sigma,\Omega_1) \subset \Omega(\sigma,\Omega_1)$ and

$$\forall \{\mu_n\} \in \sigma^{\mathbb{N}} \qquad K(\sup_n\{\mu_n\},\Omega_1) = \bigcup_n K(\mu_n,\Omega_1) , \qquad (4)$$

$$\forall \{\mu_n\} \in \sigma^{\mathbb{N}} \qquad L(\sup_n\{\mu_n\},\Omega_1) = \bigcup_n L(\mu_n,\Omega_1) , \qquad (5)$$

$$\forall \mu \in \sigma \qquad \Omega_1 \setminus K(\mu,\Omega_1) = L(1-\mu,\Omega_1) , \qquad (6)$$

$$\forall \mu \in \sigma \qquad \Omega_1 \setminus L(\mu,\Omega_1) = K(1-\mu,\Omega_1) . \qquad (7)$$

Proof: The properties (4), (5), (6) and (7) are self-ecident. Hence $\Omega(\sigma,\Omega_1)$ is closed under complementION in Ω_1 and denumerable union. Since $\Omega_1 = K(1_\Omega,\Omega_1) \in \Omega(\sigma,\Omega_1)$, the main thesis holds. Thus $K^*(\mu,\Omega_1) = L(\mu,\Omega_1) \setminus K(\mu,\Omega_1) \in \Omega(\sigma,\Omega_1)$ for every $\mu \in \sigma$ and $K^*(\sigma,\Omega_1) \subset \Omega(\sigma,\Omega_1)$. ∎

Using a fixed classical probability measure $P: \Omega(\sigma, \Omega_1) \to [0,1]$, we can describe the following motions:

Definition 3: The mapping

$$q(\cdot, \Omega_1, P): \sigma \to [0,1] \quad ,$$

such that

$$\forall\, \mu \in \sigma \qquad q(\mu, \Omega_1, P) = P(K(\mu, \Omega_1)) \quad , \tag{8}$$

is called a generalization of classical probability measure P to σ.

Definition 4: The mapping

$$\tilde{q}(\cdot, \Omega_1, P): \sigma \to [0,1] \quad ,$$

defined by

$$\forall\, \mu \in \sigma \qquad \tilde{q}(\mu, \Omega_1, P) = P(L(\mu, \Omega_1)) \quad , \tag{9}$$

is called a extended generalization of classical probability measure P to σ.

Definition 5: The fuzzy P-measure on σ is a mapping

$$p: \sigma \to \mathbb{R}^+ \cup \{0\}$$

fulfilling the next conditions:

$$\forall\, \mu \in \sigma \qquad p(\mu \vee (1-\mu)) = 1 \quad , \tag{10}$$

-if $\{\mu_n\}$ is finite or infinite sequence of pairwise W-separated fuzzy subsets (see [6]), i.e. $\mu_i \leq 1 - \mu_j$ for every i,j such that $i \neq j$, then

$$p(\sup_n \{\mu_n\}) = \sum_n p(\mu_n) \quad . \qquad [4] \tag{11}$$

Lemma 1: If the fuzzy subsets μ and ν in σ are W-separated then $K(\mu, \Omega_1) \cap K(\mu, \Omega_1) = \emptyset$.

Proof: Let us suppose that $K(\mu, \Omega_1) \cap K(\nu, \Omega_1) \neq \emptyset$. Then, for each $\omega \in K(\mu, \Omega_1) \cap K(\nu, \Omega_1)$, we have

$$\frac{1}{2} < \mu(\omega) \leqslant 1 - \nu(\omega) < \frac{1}{2} .$$

Contradiction! ■

Theorem 2: The mapping $q(\cdot, \Omega_1, P)$, defined by (8), is a fuzzy P-measure on σ iff classical probability measure P fulfils

$$\forall\, M \in K^*(\sigma, \Omega_1) \qquad P(M) = 0 . \tag{12}$$

Moreover, we have

$$q(\chi_{\Omega_2}, \Omega_1, P) = 0 . \tag{13}$$

Proof: By the Lemma 1 and (4), we get

$$q(\sup_n \{\mu_n\}, \Omega_1, P) = P(K(\sup_n \{\mu_n\}, \Omega_1)) = P(\bigcup_n K(\mu_n, \Omega_1)) =$$

$$= \sum_n P(K(\mu_n, \Omega_1)) = \sum_n q(\mu_n, \Omega_1, P)$$

for every sequence $\{\mu_n\}$ of pairwise W-separated fuzzy subsets in σ. So, (11) always holds.

Since

$$q(\mu \vee (1-\mu), \Omega_1, P) = P(K(\mu \vee (1-\mu), \Omega_1)) = P(\Omega_1 \setminus K^*(\mu, \Omega_1)) =$$
$$P(\Omega_1) - P(K^*(\mu, \Omega_1)) = 1 - P(K^*(\mu, \Omega_1))$$

for each $\mu \in \sigma$, the mapping q satisfies (10) iff (12).

Then we have

$$q(\chi_{\Omega_2}, \Omega_1, P) = P(K(\chi_{\Omega_2}, \Omega_1)) = P(\emptyset) = 0 . \; ■$$

Theorem 3: The mapping $\tilde{q}(\cdot, \Omega_1, P)$, defined by (9), is a fuzzy P-measure on σ iff (12). Furthermore, then it satisfies (13) and

$$\forall\, \mu \in \sigma \qquad \tilde{q}(\mu, \Omega_1, P) = q(\mu, \Omega_1, P) . \tag{14}$$

Proof: The identity (14) follows from (12) because

$\tilde{q}(\mu, \Omega_1, P) = P(L(\mu, \Omega_1)) = P(K(\mu, \Omega_1) \cup K^*(\mu, \Omega_1)) =$

$= P(K(\mu, \Omega_1)) + P(K^*(\mu, \Omega_1)) = q(\mu, \Omega_1, P) + P(K^*(\mu, \Omega_1))$.

So, if (12) then the mapping $\tilde{q}$ is a fuzzy P-measure on σ fulfilling (13).

Let us assume that the condition (12) does not hold. Then there exists such $\mu^* \in \sigma$ that $P(K^*(\mu^*, \Omega_1)) \neq 0$. By means of (6), we get

$\tilde{q}(\mu^*, \Omega_1, P) + \tilde{q}(1 - \mu^*, \Omega_1, P) = P(L(\mu^*, \Omega_1)) + P(L(1 - \mu^*, \Omega_1)) =$

$= P(K(\mu^*, \Omega_1) \cup K^*(\mu^*, \Omega_1)) + P(\Omega_1 \setminus K(\mu^*, \Omega_1)) =$

$= P(K(\mu^*, \Omega_1)) + P(K^*(\mu^*, \Omega_1)) + P(\Omega_1) - P(K(\mu^*, \Omega_1)) \neq P(\Omega_1) =$

$= P(L(\mu^* \vee (1 - \mu^*), \Omega_1)) = \tilde{q}(\mu^* \vee (1 - \mu^*), \Omega_1, P)$.

This fact shows that the condition (11) does not hold. The proof is complete. ■

Is any fuzzy P-measure on σ a generalization of one classical probability measure to σ ? In more special shape $(\Omega_1 = \Omega)$, this problem was given by dr Stefan Chanas on "One Day Meeting in Fuzzy Mathematics", which was held June 26, 1984 in Poznań. The solution of the above question is based on the below considerations.

Let $\bar{\mathbb{R}} = [-\infty, +\infty]$, the mapping $\varrho : \bar{\mathbb{R}}^2 \to [0,1]$ be quasi-antisymmetrical and continuous from above fuzzy relation "less or equal" (FLE) unfuzzily bounding the real line (see [2]). The FLE ϱ generates a fuzzy relation "less than" ϱ_s given by

$$\forall (x,y) \in \bar{\mathbb{R}}^2 \qquad \varrho_s(x,y) = 1 - \varrho(y,x) \ . \qquad [2] \qquad (15)$$

Then we define a fuzzy intervals (FI) as mappings $\varphi[a,b[: \bar{\mathbb{R}} \to [0,1]$ and $\varphi[a,+\infty] : \bar{\mathbb{R}} \to [0,1]$ defined, for every $(a,b) \in \bar{\mathbb{R}}^2$, by the identities

$$\forall\ x\in \overline{\mathbb{R}} \qquad \varphi[a,b[\,(x) = \varrho(a,x)\wedge\varrho_s(x,b)\ , \tag{16}$$

$$\forall\ x\in \overline{\mathbb{R}} \qquad \varphi[a,+\infty]\,(x) = \varrho(a,x)\wedge\varrho(x,+\infty)\ . \quad [3] \tag{17}$$

Among other things, for each pair $(a,b)\in\overline{\mathbb{R}}^2$, we have

$$\forall\ x\in\ ]a,+\infty] \qquad \varphi[a,+\infty]\,(x) > \tfrac{1}{2}\ , \tag{18}$$

$$\forall\ x\in [-\infty,a[\qquad \varphi[a,+\infty]\,(x) < \tfrac{1}{2}\ , \tag{19}$$

$$\forall\ x\in\ ]a,b[\qquad \varphi[a,b[\,(x) > \tfrac{1}{2}\ , \tag{20}$$

$$\forall\ x\in [-\infty,a[\,\vee\,]b,+\infty] \qquad \varphi[a,b[\,(x) < \tfrac{1}{2}\ . \tag{21}$$

More details about FI are contained in [3] . Moreover, there can be defined the following family of fuzzy subsets in $\overline{\mathbb{R}}$.

Definition 6: The smallest fuzzy σ-algebra, $\mathcal{B}_\varrho$ say, containing all FI defined by (16) or (17), is called infinite fuzzy Borel family [3] .

Theorem 4: $\mathcal{B}_\varrho$ is a soft fuzzy σ-algebra. Furthermore, each fuzzy subset $\mu\in\mathcal{B}_\varrho$ can be described by the identities

$$\mu = \mu_1 = \sup_n \{\varphi[a_n,b_n[\,\} \tag{22}$$

or

$$\mu = \mu_2 = \sup_n \{\varphi[a_n,b_n[\,\}\vee\varphi[a_0,+\infty]\ . \quad [3] \tag{23}$$

Since $\chi_{\{+\infty\}} = \varphi[+\infty,+\infty]\in\mathcal{B}_\varrho$, according to (1) $\overline{\mathbb{R}}$ can be decomposed as union $\overline{\mathbb{R}} = [-\infty,+\infty[\,\cup\,[+\infty,+\infty]$. This fact is helpful for our considerations because each classical probability measure in the real line is defined on a σ-algebra of crisp subsets in $[-\infty,+\infty[$. In accordance with (13) we ought to mark off the following class of fuzzy P-measures on $\mathcal{B}_\varrho$.

Definitions 7: The fuzzy P-measure $p: \beta_\varsigma \to \mathbb{R}^+ \cup \{0\}$, satisfying the conditions

$$p(\varphi[+\infty,+\infty]) = 0 \quad , \tag{24}$$

is called a natural fuzzy P-measure. [5]

Definition 8: The projection β_ς on $2^{[-\infty,+\infty[}$ is a mapping

$$\pi_\varsigma : \beta_\varsigma \to 2^{[-\infty,+\infty[}$$

defined by the identity

$$\forall \mu \in \beta_\varsigma \qquad \pi_\varsigma(\mu) = \begin{cases} \bigcup\limits_n [a_n,b_n[& \mu = \mu_1 \\ \bigcup\limits_n [a_n,b_n[\cup [a_0,+\infty[& \mu = \mu_2 , \end{cases} \tag{25}$$

where μ_1 and μ_2 are described respectively by (22) or (23).

Definition 9: The mapping

$$p_\alpha : \beta_\varsigma \to \{0,1\}$$

defined, for fixed $\alpha \in]-\infty,+\infty[$, by

$$\forall \mu \in \beta_\varsigma \qquad p_\alpha(\mu) = \begin{cases} 1 & \alpha \in \pi_\varsigma(\mu) \\ 0 & \alpha \notin \pi_\varsigma(\mu) \end{cases} \tag{26}$$

is called a fuzzy atomic measure on β_ς .

Lemma 2: The projection π_ς satisfies the following properties:

$$\forall \{\mu_n\} \in \beta_\varsigma^{\mathbb{N}} \qquad \pi_\varsigma(\sup_n \{\mu_n\}) = \bigcup_n \pi_\varsigma(\mu_n) \quad , \tag{27}$$

$$\forall (\mu,\nu) \in \beta_\varsigma^2 \qquad \mu \leqslant 1-\nu \Rightarrow \pi_\varsigma(\mu) \cap \pi(\nu) = \emptyset \quad , \tag{28}$$

$$\forall \mu \in \beta_\varsigma \qquad \pi_\varsigma(\mu \vee (1-\mu)) = [-\infty,+\infty[\quad . \tag{29}$$

Proof: The property (27) is obvious. If $\widetilde{\Pi}_\varsigma(\mu) \cap \widetilde{\Pi}_\varsigma(\nu) \neq \emptyset$ then there exists such pair $(a,b) \in [-\infty,+\infty[^2$ that $a<b$ and $[a,b[\subset \widetilde{\Pi}_\varsigma(\mu) \cap \widetilde{\Pi}_\varsigma(\nu)$. The conditions (18) and (20) imply: $\mu(x) > \frac{1}{2}$ and $\nu(x) > \frac{1}{2}$ for each $x \in]a,b[$. So, μ and ν are not W-separated.

Obviously, we have $\widetilde{\Pi}_\varsigma(\mu \vee (1-\mu)) \subset [-\infty,+\infty[$ for every $\mu \in \beta_\varsigma$. Let us suppose that $[-\infty,+\infty[/ \widetilde{\Pi}_\varsigma(\mu \vee (1-\mu)) \neq \emptyset$. Then there exists nonempty crisp interval $[a,b[$ such that $[a,b[\cap \widetilde{\Pi}_\varsigma(\mu \vee (1-\mu)) = \emptyset$. This fact along with (19) and (21) proofs that $\mu(x) \vee (1-\mu(x)) < \frac{1}{2}$ for each $x \in]a,b[$. Contradiction! ■

Theorem 5: The fuzzy atomic measure p_α on β_ς is a natural fuzzy P-measure on β_ς for every $\alpha \in]-\infty,+\infty[$.

Proof: The condition (10) follows from (29) and the Definition 9.

Let $\{\mu_n\}$ be a sequence of pairwise W-separated fuzzy subsets in β_ς . If $\alpha \in \widetilde{\Pi}_\varsigma(\sup_n \{\mu_n\})$ then, according to (27) and (28), there exists the unique positive integer k such that $\alpha \in \widetilde{\Pi}_\varsigma(\mu_k)$. Therefore,

$$p_\alpha(\sup_n \{\mu_n\}) = 1 = p_\alpha(\mu_k) = \sum_n p_\alpha(\mu_n) \ .$$

Using (27), we get: if $\alpha \notin \widetilde{\Pi}_\varsigma(\sup_n \{\mu_n\})$ then $\alpha \notin \widetilde{\Pi}_\varsigma(\mu_n)$ for all positive integers n . Hence

$$p_\alpha(\sup_n \{\mu_n\}) = 0 = \sum_n p_\alpha(\mu_n) \ .$$

So, the fuzzy atomic measure p_α is a fuzzy P-measure on β_ς . Since $\widetilde{\Pi}_\varsigma(\varphi[+\infty,+\infty]) = \emptyset$, the conditions (24) holds, too. ■

Definition 10: The cumulative distribution function of fuzzy P-measure $p: \beta_\xi \to \mathbb{R}^+ \cup \{0\}$ is a mapping

$$F: \bar{\mathbb{R}} \to [0,1]$$

such that

$$\forall x \in \bar{\mathbb{R}} \qquad F(x) = p\,([-\infty, x[\,). \quad [5] \tag{30}$$

Theorem 6: If a natural fuzzy P-measure $p: \beta_\xi \to \mathbb{R}^+ \cup \{0\}$ is a generalization of classical probability measure $P: \Omega(\beta_\xi, [-\infty,+\infty[\,) \to$ $\to [0,1]$ then

$$\forall x \in [-\infty,+\infty[\qquad p(\varphi[-\infty,x[\,) = P([-\infty,x[\,) = F(x) \;. \tag{31}$$

Proof: Since (21) and $\varphi[-\infty,x[\,(-\infty) = 1$ for each $x \in \bar{\mathbb{R}}$,

$$p(\varphi[-\infty,x[\,) = P(K(\varphi[-\infty,x[\,, [-\infty,+\infty[\,)) = P([-\infty,x[\,). \blacksquare$$

Moreover, the cumulative distribution function of fuzzy atomic measure p_α is given by

$$F_\alpha(x) = \begin{cases} 0 & x \leq \alpha \\ 1 & x > \alpha \end{cases} . \tag{32}$$

Let us consider the fuzzy atomic measure p_α on the soft fuzzy σ-algebra $\beta_{\hat\xi}$ generated by FLE $\hat\xi$ described as follow

$$\hat\xi(x,y) = \begin{cases} 1 & x<y \quad \text{or} \quad x=y=-\infty \quad \text{or} \quad x=y=+\infty \\ \frac{1}{2} & x=y \in \,]-\infty,+\infty[\\ 0 & x>y \end{cases}$$

According with the Theorem 6, if there exists an ordinary probability measure $P_\alpha : \Omega(\beta_{\hat\xi}, [-\infty,+\infty[\,) \to [0,1]$ such that p_α is a generalization of P_α then p_α ought to be expressed as follow

$$p_\alpha(\mu) = \int_{K(\mu,[-\infty,+\infty[)} dF = Q_\alpha(K(\mu,[-\infty,+\infty[\,))$$

for every $\mu \in \mathcal{B}_{\hat{\xi}}$. On the other side, the mapping $Q_{\alpha}(K(\cdot,[-\infty,+\infty[)) : \mathcal{B}_{\hat{\xi}} \to [0,1]$ is not a fuzzy P-measure on $\mathcal{B}_{\hat{\xi}}$ because, for $\varphi[-\infty,\alpha[\in \mathcal{B}_{\hat{\xi}}$, we have

$$Q_{\alpha}(K(\varphi[-\infty,\alpha[\vee (1-\varphi[-\infty,\alpha[), [-\infty,+\infty[)) = \int_{[-\infty,\alpha[\cup]\alpha,+\infty[} dF = 0.$$

So, there exists a natural fuzzy P-measure which is not a generalization of one classical probability measure.

In addition, let us investigate the next notions

Definition 11: The finite or infinite sequence $\{(\alpha_k, \beta_k)\}$ of pairs in $]-\infty,+\infty[\times [0,1]$ such that $\{\alpha_k\}$ is increasing and

$$\sum_k \beta_k = 1 \tag{33}$$

is called a discrete distribution.

Definition 12: Let $\{(\alpha_k, \beta_k)\}$ be a fixed discrete distribution. Then the mapping

$$d: \mathcal{B}_{\xi} \to [0,1]$$

defined as follow

$$\forall \mu \in \mathcal{B}_{\xi} \qquad d(\mu) = \sum_k \beta_k \cdot P_{\alpha_k}(\mu) \tag{34}$$

is called a discrete probability measure generated by $\{(\alpha_k, \beta_k)\}$.

Theorem 7: The discrete probability measure, generated by any discrete distribution $\{(\alpha_k, \beta_k)\}$ is a natural fuzzy P-measure on $\mathcal{B}_{\xi}$. Its cumulative distribution function is given by the identity

$$\forall x \in \mathbb{R} \qquad F(x) = 0 + \sum_{\alpha_k < x} \beta_k . \tag{35}$$

Proof: Using the Theorem 5 we have:

- for each $\mu \in \mathcal{B}_{\xi}$

$$d(\mu\vee(1-\mu)) = \sum_k \beta_k p_{\alpha_k}(\mu\vee(1-\mu)) = \sum_k \beta_k = 1;$$

- for each sequence $\{\mu_n\}$ of pairwise W-separated fuzzy subsets in $\mathcal{B}_S$

$$d(\sup_n\{\mu_n\}) = \sum_k \beta_k p_{\alpha_k}(\sup_n\{\mu_n\}) = \sum_k \beta_k \sum_n p_{\alpha_k}(\mu_n) =$$

$$= \sum_n \sum_k \beta_k p_{\alpha_k}(\mu_n) = \sum_n d(\mu_n) \quad ;$$

- $d(\varphi[+\infty,+\infty]) = \sum_k \beta_k p_{\alpha_k}(\varphi[+\infty,+\infty]) = 0$.

So, the mapping d is a natural fuzzy P-measure on $\mathcal{B}_S$.

Moreover,

$$F(x) = d(\varphi[-\infty,x[) = \sum_k \beta_k p_{\alpha_k}(\varphi[-\infty,x[) = 0 + \sum_{\alpha_k < x} \beta_k \ . \ \blacksquare$$

On the second side, the formula (35) presents any step-function $F_S : \overline{\mathbb{R}} \to [0,1]$ which fulfils:

$$F_S(-\infty) = 0 \quad , \tag{36}$$

$$F_S(+\infty) = 1 \quad , \tag{37}$$

$$\forall (x,y) \in \overline{\mathbb{R}}^2 \qquad x \leq y \Rightarrow F_S(x) \leq F_S(y) \quad , \tag{38}$$

$$\forall \{x_n\} \in \overline{\mathbb{R}}^{\mathbb{N}} \qquad \{x_n\} \uparrow x \Rightarrow \{F_S(x_n)\} \uparrow F(x) \ . \tag{39}$$

Theorem 8: If the function $F: \overline{\mathbb{R}} \to [0,1]$ satisfies (36), (37), (38) and (39) then there exists the unique natural fuzzy P-measure on $\mathcal{B}_S$ fulfilling (30) [5] .

Therefore, we have the finishing conclusion.

Theorem 9: The discrete probability measure, generated by $\{(\alpha_k, \beta_k)\}$, is the unique natural fuzzy P-measure on $\mathcal{B}_S$ such that its cumulative distribution function is the step-function given by (35).

Acknowledgments

This paper contains the result of the work in the Seminar on Fuzzy and Interval Mathematics directed by Prof. dr hab. Jerzy Albrycht.

References:

[1] S.Khalili, Fuzzy Measures and Mappings, J.Math.Anal.Appl. 68 (1979), 92-99.

[2] K.Piasecki, On any Class of Fuzzy Preference Relation in Real Line - Part II, BUSEFAL 21 (1985), 82-92.

[3] K.Piasecki, On Interval Defined by Fuzzy Preference Relation, BUSEFAL 22 (1985), 58-65.

[4] K.Piasecki, Probability of Fuzzy Events Defined as Denumerable Additivity Measure, Fuzzy Sets and Systems 17 (1985), 1-14.

[5] K.Piasecki, Fuzzy P-measure on the Real Line, (to appear in Fuzzy Sets and Systems).

[6] K.Piasecki, New Concept of Separated Fuzzy Subsets, Proc. of the Polish Symposium on Interval and Fuzzy Mathematics, Ed. by J.Albrycht and H.Wiśniewski (1985), 193-196.