The Convergence of Measurable Function Sequences on the Possibility Measure Space

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Abstract

In this paper, we shall introduce the concepts of almost and pseudo-almost on the possibility measure space, and study the convergence properties of measurable function sequences.

1. Possibility Measure Space

Definition1.1. Let Ω be a nonempty set, $\mathcal{P}=\mathcal{P}(\Omega)$ be the power set of Ω. A mapping $\pi: \mathcal{P} \to [0, 1]$ is called the possibility measure, if it satisfies the following conditions:

(P1) $\pi(\phi)=0$;

(P2) $\forall \{A_t | t \in T\} \subset \mathcal{P}$, $\pi(\bigcup_{t\in T} A_t)=\sup_{t\in T}\pi(A_t)$, where T is an arbitrary index set.

We call (P2) "Fuzzy-additivity". Another equivalent definition is the following.

Definition1.2. If $f: \Omega \to [0, 1]$, we define the set function as follows,

$\pi(A)=\sup_{\omega\in A} f(\omega)$, and $\sup_{\omega\in\phi} f(\omega)=0$, $\sup_{t\in\phi} \pi(A_t)=0$ are made, where $A \in \mathcal{P}$.

The π is a possibility measure on Ω. We call f the density of π.

Property1.1. 1) The possibility measure possesses the monotonicity: $\forall$ A, B $\in \mathcal{P}$, $A\subset B \Rightarrow \pi(A) \leq \pi(B)$;

2) The possibility measure possesses the continuity from below: $\forall A_i \in \mathcal{P}$ (i=1, 2, $\cdots$), $A_i \uparrow \Rightarrow \pi(\lim_{n\to\infty} A_n)=\lim_{n\to\infty}\pi(A_n)$;

3) The possibility measure is uniformly autocontinuous, and therefore, autocontinuous, null-additive ([1]).

Because the possibility measure generally does not possess the continuity from above, it is not necessarily a fuzzy measure([3]).

The triple $(\Omega, \wp, \pi)$ will be refered to as a possibility measure space. Any real-valued functions defined on Ω are $\wp$-measurable.

In the following, we always assume that $(\Omega, \wp, \pi)$ is a possibility measure space, and X, X_n, $n=1, 2, \cdots$, are real-valued functions on Ω.

Definition1.3. Let $A \in \wp$, $p(w)$ be a proposition on A. If there exists $E \in \wp$ with $\pi(E)=0$, such that $p(w)$ is true on A-E, then we say "p(w) is true almost everywhere on A"; If there exists $F \in \wp$ with $\pi(A-F)=\pi(A)$, such that $p(w)$ is true on A-F, then we say "p(w) is true pseudo-almost everywhere on A".

Especially, with sequences $\{X_n\}$ converges to X instead of proposition $p(w)$,we may have the definitions of "almost everywhere convergence" and "pseudo-almost everywhere convergence". They are denoted by $X_n \xrightarrow[A]{a.e.} X$ and $X_n \xrightarrow[A]{p.a.e.} X$ respectively.

Proposition1.1.Let $A \in \wp$, $p(w)$ be a proposition on A. If $p(w)$ is true almost everywhere on A, then it true also pseudo-almost everywhere on A.

Proof. By using definition1.3 and the null-additivity, it is easy to obtain this conclusion. ∎

Definition1.4. Let $A \in \wp$. If there exists $\{E_n\} \subset \wp$ with $\lim_{n\to\infty} \pi(E_n)=0$, such that $\{X_n\}$ converges to X uniformly on A-E ,n=1,2,···, then we say "$\{X_n\}$ converges to X almost uniformly on A", and denote it by $X_n \xrightarrow[A]{a.u.} X$; If there exists $\{F_n\} \subset \wp$ with $\lim_{n\to\infty} \pi(A-F_n)=\pi(A)$, such that $\{X_n\}$ converges to X uniformly on $A-F_n$, $n=1,2,\cdots$, then we say "$\{X_n\}$ converges to X pseudo-almost everywhere on A", and denote it by $X_n \xrightarrow[A]{p.a.u.} X$.

Definition1.5. Let $A\in\wp$. If there exists $E\in\wp$ with $\pi(E)=0$, such that $\{X_n\}$ converges to X uniformly on A, then we say "$\{X_n\}$ converges to X almost everywhere uniformly on A", and denote it by $X_n\xrightarrow[A]{a.e.u.}X$; If there exists $F\in\wp$ with $\pi(A-F)=\pi(A)$, such that $\{X_n\}$ converges to X uniformly on A-F, then we say "$\{X_n\}$ converges to X pseudo-almost everywhere uniformly on A", and denote it by $X_n\xrightarrow[A]{p.a.e.u.}X$.

Definition1.6. Let $A\in\wp$. If $\lim\limits_{n\to\infty}\pi(\{|X_n-X|\geq\varepsilon\}\cap A)=0$ for any given $\varepsilon>0$, then we say $\{X_n\}$ converges in possibility measure π to X on A, and denote it by $X_n\xrightarrow[A]{\pi}X$; If $\lim\limits_{n\to\infty}\pi(\{|X_n-X|<\varepsilon\}\cap A)=\pi(A)$ for any given $\varepsilon>0$, then we say $\{X_n\}$ converges pseudo-in possibility measure π to X on A, and denote it by $X_n\xrightarrow[A]{p\pi}X$.

Proposition1.2. If a real-valued function sequence converges in some meaning above, then it must pseudo- converge in the same meaning.

From the following examples, we can see that the converse result of proposition1.2 is generally not true.

Example1.1. consider $\Omega=R^1=(-\infty,+\infty)$, and the measure π defined by $f(\omega)=\begin{cases}1, & \omega\geq 0\\ 0, & \omega<0\end{cases}$, denote $E_k=(-\infty,\frac{1}{k})$, $S_n(\omega)=\frac{\omega}{1+n^2\omega^2}$, $S(\omega)=\begin{cases}0, & \omega>0\\ 1, & \omega\leq 0\end{cases}$.

Clearly, $\{S_n(\omega)\}$ converges uniformly to constant 0. Therefore, $\{S_n(\omega)\}$ converges uniformly to 0 on $(0,+\infty)$. Furthermore, it converges uniformly to $S(\omega)$ uniformly on $\Omega-E_k$. And since $\pi(\Omega-E_k)=\pi(\Omega)$, we have: $S_n(\omega)\xrightarrow[\Omega]{p.a.u.}S(\omega)$; For $S_n(0)=0\not\to 1=S(0)$ and $\pi(\{0\})=1$, consequently, $S_n(\omega)\overset{a.u.}{\underset{\Omega}{\not\to}}S(\omega)$.

Example1.2. Consider $\Omega=\{a,b,c\}$, and the measure π defined by its density $f(\omega)=\begin{cases}0, & \omega=a\\ 1, & \omega=b,c\end{cases}$. Let $X_n(\omega)=\begin{cases}1-\frac{1}{n}, & \omega=a,c\\ 0, & \omega=b\end{cases}$, $X(\omega)\equiv 1$.

From $\lim\limits_{n\to\infty}\pi(\{|X_n-X|<\varepsilon\})=\pi(\{a,c\})=1=\pi(\Omega)$, and $\lim\limits_{n\to\infty}\pi(\{|X_n-X|\geq\varepsilon\})=\pi(\{b\})=1\neq 0$ for any given $\varepsilon>0$, we obtain

$$X_n\xrightarrow[\Omega]{p\pi}X,\quad X_n\overset{\pi}{\underset{\Omega}{\not\to}}X.$$

2. The relation among convergences

Theorem2.1. Let $A \in \mathcal{O}$, then

$$X_n \xrightarrow[A]{a.e.u.} X \Rightarrow X_n \xrightarrow[A]{a.u.} X \Rightarrow X_n \xrightarrow[A]{a.e.} X \tag{2.1}$$

$$X_n \xrightarrow[A]{p.a.e.u.} X \Rightarrow X_n \xrightarrow[A]{p.a.u.} X \Rightarrow X_n \xrightarrow[A]{p.a.e.} X \tag{2.2}$$

$$X_n \xrightarrow[A]{a.u.} X \Rightarrow X_n \xrightarrow[A]{p.a.e.u.} X \tag{2.3}$$

Proof. From the definitions and by using property1.1, it is not difficult to obtain the conclusions (2.1) and (2.2). We now prove the last conclusion (2.3).

As $X_n \xrightarrow[A]{a.u.} X$, there exists $\{E_k\}$, $E_k \subset A, k=1, 2, \cdots$, and $\lim\limits_{k\to\infty} \pi(E_k)=0$, such that $\{X_n\}$ converges to X uniformly on $A-E_k$. If $\pi(A)=0$, then $X_n \xrightarrow[A]{p.a.e.u.} X$ evidently; Let $\pi(A)$ 0. For $\lim\limits_{k\to\infty} \pi(E_k)=0$, there exists k, $\pi(E_k) < \pi(A)$ and so we have

$$\pi(A)=\pi[(A-E_k)\cup E_k]= \pi(A-E_k)\cup\pi(E_k)=\pi(A-E_k).$$

Namely, $\{X_n\}$ converges to X uniformly on $A-E_k$, i.e. $X_n \xrightarrow[A]{p.a.e.u.} X$.

The converse conclusions which correspond to (2.1), (2.2), and (2.3) are generally not true.

Theorem2.2. Let $A \in \mathcal{O}$, then

$$X_n \xrightarrow[A]{a.u.} X \Longleftrightarrow X_n \xrightarrow[A]{\pi} X \tag{2.4}$$

Proof. At first, Let $X_n \xrightarrow[A]{\pi} X$. For every m, m=1, 2, $\cdots$, there exists K_m respectively, such that $\pi(\{|X_k-X| \geq \frac{1}{m}\} \cap A) < \frac{1}{m}$ as $k \geq K_m$. Now we suppose that K_m be increasing about m, which does not bring any loss of generality. If we denote

$$E_n=\bigcup_{m\geq n} \bigcup_{k\geq K_m} (\{|X_k-X| \geq \frac{1}{m}\} \cap A),$$

then

$$\pi(E_n)=\sup_{k\geq K_m, m\geq n} \pi(\{|X_k-X| \geq \frac{1}{m}\} \cap A) \leq \frac{1}{n}.$$

It is easy to show that $\{X_k\}$ converges to X uniformly on $A-E_n$. In fact, for arbitrary given $\varepsilon>0$, taking $m > \frac{1}{\varepsilon} \vee n$, we have $\omega \notin \{|X_k-X| \geq \frac{1}{m}\}$, as $k \geq K_m$. It follows $|X_k(\omega)-X(\omega)| < \varepsilon$, for every $\omega \in A-E_n$. Therefore, $X_n \xrightarrow[A]{a.u.} X$.

Conversely, it follows, from the assumption, that for any given $\delta>0$, there exists A_δ with $\pi(A_\delta)<\delta$, $A_\delta \subset A$, such that for every given $\varepsilon>0$, there exists a natural number $N(\varepsilon,\delta)$ and we have $|X_k(\omega)-X(\omega)|<\varepsilon$ for all $\omega \in A-A_\delta$ as $k \geqslant N(\varepsilon,\delta)$. Furthermore,

$$\begin{aligned}&\pi(\{\omega \mid |X_k(\omega)-X(\omega)| \geqslant \varepsilon\} \cap A)\\ &=\pi(\{\omega \mid |X_k(\omega)-X(\omega)| \geqslant \varepsilon\} \cap A_\delta)\\ &\leqslant \delta \vee 0\\ &=\delta\end{aligned}$$

And therefore, for arbitrary given $\varepsilon>0$, we have

$$\lim_{n\to\infty}\pi(\{\omega \mid |X_k-X| \geqslant \varepsilon\} \cap A)=0,$$

namely $X_n \xrightarrow[A]{\pi} X$. The proof is completed.

The following conclusion(2.5) is stronger formally than Riesz's theorem in classical measure theory; and the conclusion(2.6) is similar to the Lebesgue's theorem in classical measure theory.

Theorem2.3. Let $A \in \mathcal{P}$, then

$$X_n \xrightarrow[A]{\pi} X \Rightarrow X_n \xrightarrow[A]{a.e.} X \tag{2.5}$$

$$X_n \xrightarrow[A]{p.a.e.} X \Rightarrow X_n \xrightarrow[A]{p.\pi} X \tag{2.6}$$

Proof. Note that (2.4) and (2.1), it is easy to prove (2.5). we now turn to the proof of (2.6).

Since $X_n \xrightarrow[A]{p.a.e.} X$, there exists set F with $F \subset A$, $\pi(A-F)=\pi(A)$, such that $\lim_{n\to\infty} X_n(\omega)=X(\omega)$ for all $\omega \in A-F$. That is, for any given $\varepsilon>0$, $\omega \in A-F$, there exists a natural number $N(\omega)$. such that $|X_n(\omega)-X(\omega)|<\varepsilon$ as $n \geqslant N(\omega)$. Denote $A_n=\{\omega \mid N(\omega)\leqslant n\} \cap (A-F)$, and for $A_n \subset \{|X_n-X|<\varepsilon\} \cap A$, we have

$$\pi(A) \geqslant \pi(\{|X_n-X|<\varepsilon\} \cap A) \geqslant \pi(A_n) \to \pi(A-F)=\pi(A).$$

And hence, $X_n \xrightarrow[A]{p.\pi} X$.

Corollary2.1. If π is continuous from above, then

$$X_n \xrightarrow[A]{a.u.} X \rightleftharpoons X_n \xrightarrow[A]{\pi} X \rightleftharpoons X_n \xrightarrow[A]{a.e.} X .$$

Proof. From $X_n \xrightarrow{a.e.} X \Rightarrow X_n \xrightarrow[A]{\pi} X$ ([1]) and by using the conclusions (2.4) and (2.5), the corollary may be followed. ∎

Corollary2.[illegible]. Let $A \in \mathcal{C}$, then

$$X_n \xrightarrow[A]{p.a.u.} X \Rightarrow X_n \xrightarrow[A]{p.\pi} X .$$

Proof. It is followed from the conclusions (2.2) and (2.5). ∎

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References

[1] WANG Zhenyuan, The autocontinuity of set function and fuzzy integral, J. Math. Anal. Appl. 99(1984) 195-218.

[2] WANG Zhenyuan, Asymptotic structural characteristics of fuzzy measure and their applications, to appear in Fuzzy Sets and Systems.

[3] M. L. Puri and D. Rulescu, A possibility measure is not a fuzzy measure, Fuzzy Sets and Systems 7(1982) 311-313.

[4] L.A. Zadeh, Fuzzy sets as basis for a theory of possibility, Fuzzy Sets and Systems 1(1978) 3-28.

[5] M. Loéve, Probability Theory I , 4th. edition.