SEMI-LATTICE STRUCTURE OF ALL EXTENSIONS OF POSSIBILITY MEASURE AND CONSONANT BELIEF FUNCTION

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In this paper, we give a necessary and sufficient condition for that a mapping μ from an arbitrary nonempty class $\mathcal{E}$ of subsets of X into the unit interval [0,1] can be extended to a possibility measure on the power set $\mathcal{P}(X)$ of X, and analogously, a necessary and sufficient condition for extending to a consonant belief function. We also discuss the algebraic structure of all of extensions.

Keywords: possibility measure, belief function.

INTRODUCTION

Possibility measure and consonant belief function may be used in system research as expert's subjective evaluation to a family of events. In some previous research,[1,2,6] to determine a possibility measure or a consonant belief function, it requires enough fine and quite complete knowledge. But this requirement isn't always realizable. The extension theory of possibility measure and consonant belief function provides theoretical basis and practical method for profiting fully from rough and incomplete information. Therefore, the results in this paper are useful for information processing.

Throughout this paper, let X be a nonempty set, $\mathcal{P}(X)$ be the power set of X, $\mathcal{E}$ be an arbitrarily given nonempty subset of $\mathcal{P}(X)$. We make the conventions: $\bigcup_{\phi}\{\cdot\} = \phi$, $\bigcap_{\phi}\{\cdot\} = X$, $\sup_{t\in\phi}\{a_t \mid a_t \in [0,1]\} = 0$, $\inf_{t\in\phi}\{a_t \mid a_t \in [0,1]\} = 1$.

EXTENSION OF POSSIBILITY MEASURE

Definition 1 A mapping $\pi : \mathcal{P}(X) \longrightarrow [0,1]$ is called the possibility measure on X, if $\forall \{A_t \mid t \in T\} \subset \mathcal{P}(X)$,

$$\pi(\bigcup_{t \in T} A_t) = \operatorname{Sup}_{t \in T} \pi(A_t),$$

where T is arbitrary index set.

Evidently, the possibility measure π is nondecreasing, and $\pi(\phi) = 0$.

This definition is weaker than Zadeh's one (abandoned the requirement $\pi(X) = 1$).

Definition 2 A mapping $\mu : \mathcal{E} \longrightarrow [0,1]$ is called P-consistent, if $\forall \{A_t \mid t \in T\} \subset \mathcal{E}$, $\forall A \in \mathcal{E}$,

$$A \subset \bigcup_{t \in T} A_t \Longrightarrow \mu(A) \leq \operatorname{Sup}_{t \in T} \mu(A_t),$$

where T is arbitrary index set.

Theorem 1 A mapping $\mu : \mathcal{E} \longrightarrow [0,1]$ can be extended to a possibility measure on X, if and only if μ is P-consistent.

Proof. Necessity: Let μ can be extended to a possibility measure on X. Noting that π is nondecreasing, for any $\{A_t \mid t \in T\} \subset \mathcal{E}$ and any $A \in \mathcal{E}$, if $A \subset \bigcup_{t \in T} A_t$, then we have

$$\mu(A) = \pi(A) \leq \pi(\bigcup_{t \in T} A_t) = \operatorname{Sup}_{t \in T} \pi(A_t)$$
$$= \operatorname{Sup}_{t \in T} \mu(A_t).$$

Sufficiency: Let μ be P-consistent. We define a mapping

$$\pi : \mathcal{P}(X) \longrightarrow [0,1]$$
$$B \longmapsto \operatorname{Sup}_{x \in B} \operatorname{Inf}_{x \in A \in \mathcal{E}} \mu(A). \qquad (*)$$

It is easy to see that π is a possibility measure on X. The following shows that this mapping π is an extension of μ on $\mathcal{E}$, i.e., $\forall B \in \mathcal{E}$, there holds $\pi(B) = \mu(B)$. Take $B \in \mathcal{E}$ arbitrarily. On one hand, from the expression (*), since $B \in \{A \mid x \in A \in \mathcal{E}\}$ when $x \in B$, we have

$$\pi(B) \leq \operatorname{Sup}_{x \in B} \mu(B) = \mu(B).$$

On the other hand, arbitrarily given $\varepsilon > 0$, $\forall x \in B$, $\exists A_x \in \mathcal{E}$, such that $x \in A_x$, and

$$\operatorname*{Inf}_{x \in A \in \mathcal{C}} \mu(A) \geq \mu(A_x) - \varepsilon.$$

Since $\bigcup_{x \in B} A_x \supset B$, by using the P-consistency of μ , we have

$$\operatorname*{Sup}_{x \in B} \mu(A_x) \geq \mu(B),$$

and therefore

$$\pi(B) = \operatorname*{Sup}_{x \in B} \operatorname*{Inf}_{x \in A \in \mathcal{C}} \mu(A)$$

$$\geq \operatorname*{Sup}_{x \in B} [\mu(A_x) - \varepsilon]$$

$$\geq \mu(B) - \varepsilon.$$

Because ε may close to zero arbitrarily, we obtain

$$\pi(B) \geq \mu(B).$$

Consequently,

$$\pi(B) = \mu(B). \quad |$$

EXTENSION OF CONSONANT BELIEF FUNCTION

Definition 3 A mapping $\beta : \mathcal{P}(X) \rightarrow [0,1]$ is called the consonant belief function on X, if $\forall \{A_t \mid t \in T\} \subset \mathcal{P}(X)$,

$$\beta(\bigcap_{t \in T} A_t) = \operatorname*{Inf}_{t \in T} \beta(A_t),$$

where T is arbitrary index set.

Evidently, the consonant belief function β is non-decreasing, and $\beta(X) = 1$.

This definition is more general than that one given by Banon [1]. Banon's definition requires that X is finite and $\beta(\phi) = 0$.

Definition 4 A mapping $\mu : \mathcal{C} \rightarrow [0,1]$ is called B-consistent, if $\forall \{A_t \mid t \in T\} \subset \mathcal{C}$, $\forall A \in \mathcal{C}$,

$$A \supset \bigcap_{t \in T} A_t \Longrightarrow \mu(A) \geq \operatorname*{Inf}_{t \in T} \mu(A_t),$$

where T is arbitrary index set.

Theorem 2 A mapping $\mu : \mathcal{C} \rightarrow [0,1]$ can be extended to a consonant belief function on X, if and only if μ is

B-consistent.

Proof. It is analogous to the proof of Theorem 1. We here only point out that when μ is B-consistent, the mapping β given by

$$\beta : \mathcal{P}(X) \longrightarrow [0,1]$$
$$B \longmapsto \operatorname*{Inf}_{x \in B} \ \operatorname*{Sup}_{x \in A \in \mathcal{E}} \mu(A) \qquad (**)$$

is a consonant belief function on X, and it is an extension of μ on $\mathcal{E}$. ∎

SEMI-LATTICE STRUCTURE OF ALL OF EXTENSION

There are some examples showing that the possibility measure extension onto $\mathcal{P}(X)$ for a mapping $\mu : \mathcal{E} \rightarrow [0,1]$ satisfying the P-consistency may be not unique. Analogously, the consonant belief function extension onto $\mathcal{P}(X)$ for a mapping $\mu : \mathcal{E} \rightarrow [0,1]$ satisfying the B-consistency may be not unique. Denote all of possibility measure extensions and all of consonant belief function extensions of μ by $\mathcal{E}_\pi(\mu)$ and $\mathcal{E}_\beta(\mu)$ respectively. From Theorem 1 and Theorem 2, $\mathcal{E}_\pi(\mu)$ is nonempty when μ is P-consistent, and $\mathcal{E}_\beta(\mu)$ is nonempty when μ is B-consistent.

Arbitrarily given two mapping $\mu_1 : \mathcal{P}(X) \longrightarrow [0,1]$ and $\mu_2 : \mathcal{P}(X) \rightarrow [0,1]$, if we define relation "$\leq$" as follows:

$$\mu_1 \leq \mu_2 \Longleftrightarrow \mu_1(A) \leq \mu_2(A) \quad \text{for every } A \in \mathcal{P}(X),$$

then "$\leq$" is a partial order on $\mathcal{E}_\pi(\mu)$ (analogously, on $\mathcal{E}_\beta(\mu)$). Furthermore, denote $\bar{\mu} = \operatorname{Sup}\{\mu_1, \mu_2\}$ and $\underline{\mu} = \operatorname{Inf}\{\mu_1, \mu_2\}$, then we have

$$\bar{\mu}(A) = \mu_1(A) \vee \mu_2(A),$$
$$\underline{\mu}(A) = \mu_1(A) \wedge \mu_2(A), \ \forall A \in \mathcal{P}(X).$$

Theorem 3 ($\mathcal{E}_\pi(\mu)$, $\leq$) is an upper semi-lattice, and the extension π given by the expression (*) is the maximum element of ($\mathcal{E}_\pi(\mu)$, $\leq$).

Proof. If π_1 and π_2 are possibility measures on X, then so is their supremum. Therefore, $(\mathcal{E}_{\pi}(\mu), \leq)$ is an upper semi-lattice. Now we turn to show the second conclusion of this theorem. Let π be the possibility measure extension of μ given by the expression (*). Arbitrarily given $\pi' \in \mathcal{E}_{\pi}(\mu)$, since for any $A \in \mathcal{C}$ and any singleton $\{x\}$ satisfying $x \in A \in \mathcal{C}$, we have

$$\pi'(\{x\}) \leq \pi'(A) = \mu(A),$$

therefore

$$\pi'(\{x\}) \leq \inf_{x \in A \in \mathcal{C}} \mu(A).$$

Consequently, $\forall B \in \mathcal{P}(X)$

$$\pi(B) = \sup_{x \in B} \inf_{x \in A \in \mathcal{C}} \mu(A) \geq \sup_{x \in B} \pi'(\{x\}) = \pi'(B).$$

That is to say, π is the maximum element of $(\mathcal{E}_{\pi}(\mu), \leq)$. ▮

Theorem 4 $(\mathcal{E}_{\beta}(\mu), \leq)$ is a lower semi-lattice, and the extension β given by the expression (**) is the minimum element of $(\mathcal{E}_{\beta}(\mu), \leq)$.

Proof. It is analogous to the proof of Theorem 3. ▮

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