FUZZY ALGEBRAIC SYSTEM

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1. INTRODUCTION

Since L.A. Zadeh proved fuzzy set theory, men have powerful interest for the method to fuzzify each mathematical notions. The works about the research of fuzzy algebra* are it's important part.

A. Rosenfeld proved concept of fuzzy group [1].

D. H. Fester and Wu Wangming etc. also have good words on this aspect.

But at present the fuzzy algebra (e.g. fuzzy group, fuzzy ring) is separately completed and one by one defined. These fuzzy methods are both mechanical and lack reasonable explanation. They do not present the general character that fuzzification is a common concept. Thus men naturally provide such problem, that is, can we commonly fuzzy the algebra?

This paper gives a common definition of fuzzy algebra and we prove that a fuzzy subsets on an algebra A is a fuzzy algebra iff for $\forall$ f holds

* In this paper will use algebra to express the algebraic system.

$$S(f(x_1,\cdots,x_n)) \geq \bigwedge_{i=1}^{n} S(x_i) \qquad (1)$$

where f is an operation of A.

This definition is sum of the present definition of each fuzzy algebra.

2. DEFINITIONS AND THEOREMS

Symbols in this paper are same with [2].

Definition 1. Let A be a nonfuzzy algebra defined by operation's set Ω, s is a subset of A. If for any operation f in Ω, s is closed, then s is called subalgebra of A.

Remark. Here we do not emphasize that s is no empty, i.e. we regard the empty set ϕ as subalgebra of any algebra.

Definition 2. Let s be a fuzzy subset of an algebra A, if for $\forall \lambda \in I$ the λ-cut S_λ is a subalgebra of A, then S is called a fuzzy algebra of A.

Theorem 1. A fuzzy set s on A is a fuzzy algebra iff there exists a decreasing chain H such

$$S = \bigcup_{\lambda \in I} \lambda H(\lambda) \qquad (1)$$

where $H = \{ H(\lambda) \mid \lambda \in I,\ H(\lambda)$ is subalgebra of $A \}$

Proof. If s is a fuzzy subalgebra. We let $H = \{ S_\lambda \mid \lambda \in I \}$. By Definition 2 the necessary condition of theorem is completed.

On the other hand, if (1) holds, by [3] for $\forall \lambda \in I$ we have

$$S_\lambda = \bigcap_{\alpha < \lambda} H(\alpha) \qquad (2)$$

Because H (α) is subalgebra of A, then H(α) is closed for operation's set Ω, thus S_λ is the same as H(α). By Definition 1 S_λ is subalgebra of A. By Definition 2 s is a fuzzy subalgebra.

Theorem 2. $f(x_1,\ldots,x_n)$ is used represent a n-ary algebraic operation of an algebra A, then a fuzzy set on A is a fuzzy subalgebra on A iff for $\forall f \in \Omega$ and $x_i \in A$ holds

$$S(f(x_1,\cdots,x_n)) \geq \bigwedge_{i=1}^{n} S(x_i) \qquad (3)$$

Proof. First we shall show that, (3) holds for nonfuzzy case. If s is closed for $f \in \Omega$, then from $x_i \in S$ we can write

$$f(x_1,\cdots,x_n) \in S$$

i.e. since $S(x_i) = 1$ follows $\bigwedge_{i=1}^{n} S(x_i) = 1$, and we can conclude that

$$S(f(x_1,\cdots x_n)) = 1$$

clearly in this case (3) is holds.

On the other hand if exists $x_i \notin S$, $\bigwedge_{i=1}^{n} S(x_i) = 0$, then (3) also holds.

If s is a fuzzy subalgebra of A, by theorem 1 there exists a decreasing chain H such that

$$S = \bigcup_{\lambda \in I} \lambda H(\lambda)$$

Thus for $\forall\ x_i \in A$ holds

$$S(f(x_1, \cdots, x_n)) = \bigvee_{\lambda \in I} \lambda \wedge H(\lambda)(f(x_1, \cdots, x_n)),$$

and

$$\bigvee_{\lambda \in I} \lambda \wedge H(\lambda)(f(x_1, \cdots, x_n)) \geq \bigvee_{\lambda \in I} \lambda \wedge \left(\bigwedge_{i=1}^{n} H(\lambda)(x_i) \right).$$

by [2] we have

$$\bigvee_{\lambda \in I} \lambda \wedge \left(\bigwedge_{i=1}^{n} H(\lambda)(x_i) \right) = \bigwedge_{i=1}^{n} \left(\bigvee_{\lambda \in I} \lambda \wedge H(\lambda)(x_i) \right) = \bigwedge_{i=1}^{n} S(x_i).$$

Following is the proof of sufficient condition of the theorem.

If (3) holds and $x_i \in S_\lambda$, $S(x_i) \geq \lambda$, then for $\forall\ f \in \Omega$ we can write

$$S(f(x_1, \cdots, x_n)) \geq \bigwedge_{i=1}^{n} S(x_i) \geq \lambda,$$

thus for $\forall\ f \in \Omega$, $f(x_1, \cdots x_n) \in S_\lambda$,

i.e. S_λ is a subalgebra. By definition 2 the proof is completed.

3. EXAMPLES

We shall give examples to fuzzify several main algebra.

Example 1. The group A has two operations. i.e. $f(a) = a^{-1}$ and $g(a,b) = ab$, by theorem 2 we can immediately get that a fuzzy set s of group A is a fuzzy subgroup iff

$$s(a^{-1}) \geq s(a),$$

$$s(a \cdot b) \geq s(a) \wedge s(b).$$

Example 2. Ring R has three operations, i.e.

$$f(a) = -a,$$
$$g(a,b) = a+b,$$
$$h(a,b) = ab.$$

Such that, a fuzzy subset s of ring R is a fuzzy subring iff for $\forall$ a, b R holds

$$s(-a) \geq s(a),$$

$$s(a+b) \geq s(a) \wedge s(b),$$

$$s(ab) \geq s(a) \wedge s(b).$$

Example 3. Similarly, we get that a fuzzy subset s of a lattice L is a fuzzy lattice iff for $\forall\ a, b \in L$ holds

$$s(a \wedge b) \geq s(a) \wedge s(b),$$

$$s(a \vee b) \geq s(a) \wedge s(b).$$

In other paper all above examples given is separately.

REFERENCES

(1) A. Rosenfeld. "Fuzzy groups". J. Math. Anal. and Appl. 35 (1971).

(2) Liu Wenbin "A common method to fuzzify nonfuzzy concept".

(3) Luo Changzhong "Fuzzy set and set-embedding". Fuzzy Mathematics (1984) (4). China