# AN ALGORITHM FOR FUZZY CLUSTERING BASED ON FUZZY RELATIONS

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ABSTRACT

In this paper, some properties of the fuzzy equivalence matrices are discussed. On this basis, we propose a simple and convenient method of hierarchical clustering by a fuzzy similar matrix $\mathbb{R}$ on the patterns set X.

KEYWORDS: Fuzzy equivalence matrix, Equivalence level, Equivalence point.

## 1. INTRODUCTION

S. Tamura et al. [ 1 ] described a hierarchical clustering scheme based on a fuzzy equivalence relation on a patterns set X. To start with, we give a fuzzy relation $\mathbb{R}$ on X by calculating the coefficients of similarty or the distances of each pair of patterns that is taken from the population of patterns to be classified. A fuzzy relation matrix $\mathbb{R}=[r_{ij}]_{n\times n}$ may express a fuzzy relation when X is a finite set. $\mathbb{R}$ is called a similar matrix if it satisfies the following two conditions,

(1°). reflexivity: $r_{ii}=1$ ,

(2°). symmetry: $r_{ij}=r_{ji}$ ,

where $0\leqslant r_{ij}\leqslant 1$, $i,j=1,2,\ldots,n$. A fuzzy matrix $\mathbb{R}$ is called an equivalence matrix if it is similar and satisfies the condition

(3°). transitivity: $\mathbb{R}\circ\mathbb{R}\leqslant\mathbb{R}$,

where $\mathbb{R}\circ\mathbb{R}=A \triangleq [a_{ij}] \Leftrightarrow a_{ij}=\bigvee_{l=1}^{n}(r_{il}\wedge r_{lj})$. (1)

Let $\mathbb{R}$ be an equivalence matrix. Because $\mathbb{R}$ is symmetric, we have

$$\mathbb{R}\circ\mathbb{R}=A \Leftrightarrow a_{ij}=\bigvee_{l=1}^{n}(r_{il}\wedge r_{jl}) \quad (2)$$

and $\mathbb{R}\circ\mathbb{R}\geqslant\mathbb{R}$, thus the condition (3°) may be replaced by

(3°)* $\mathbb{R}\circ\mathbb{R}=\mathbb{R}$ . (3)

Using an equivalence relation on the patterns set X, we can classify the present population of the patterns into some classes. However, in many experiments of the classification, the fuzzy relation matrix $\mathbb{R}$ obtained is a similar matrix, not an equivalence matrix. An improvement is to compute the transitive closure $\mathbb{R}^*$ for the fuzzy similar matrix $\mathbb{R}$. It had been proved that transitive closure of a fuzzy similar matrix $\mathbb{R}$ can be obtained by calculating $\mathbb{R}^{2^k}$ in finite steps. Obviously, the computing process is complicated when the order of $\mathbb{R}$ is high.

A great amout of the works have been done on the basis of Tamura's scheme, such as Dunn [2] proposed a method of maximal spanning trees and Zhao ruhuai [4] a net-making method. In this paper, we will propose a simple and convenient method of hierarchical clustering by fuzzy similar matrix $\mathbb{R}$ on the patterns set $X$ . It is shown that procedure is superior to algorithms of [1], [2], [4] with regard to both computing time and storage requirements.

## 2. PROPERTIES OF FUZZY EQUIVALENCE MATRICES

We suppose that fuzzy relation matrix $\mathbb{R}$ is similar through the paper.

Definition 2.1 The entry $r_{ij}$ of a fuzzy similar matrix $\mathbb{R}=[r_{ij}]_{n\times n}$ is called a equivalence level, if $r_{ij}$ ( $i \neq j$ ) is a maximal element in the $i$-th row of $\mathbb{R}$ except the diagonal element $r_{ii}$ .

Definition 2.2 Let $k\times k$ $(k\leq n)$ matrix $\mathbb{R}_{\langle k\rangle}$ be a submatrix of the $n\times n$ matrix $\mathbb{R}$, where

$$\mathbb{R} = \begin{bmatrix} r_{11} & r_{12}\cdots r_{1n} \\ r_{21} & r_{22}\cdots r_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ r_{n1} & r_{n2}\cdots r_{nn} \end{bmatrix} \quad \text{and} \quad \mathbb{R}_{\langle k\rangle} = \begin{bmatrix} r_{11} & r_{12}\cdots r_{1k} \\ r_{21} & r_{22}\cdots r_{2k} \\ \vdots & \vdots \quad\quad \vdots \\ r_{k1} & r_{k2}\cdots r_{kk} \end{bmatrix} ,$$

we call $\mathbb{R}_{\langle k\rangle}$ a main submatrix of $\mathbb{R}$.

Proposition 2.1 If $n\times n$ fuzzy matrix $\mathbb{R}$ is equivalent, then each order main submatrices $\mathbb{R}_{\langle 1\rangle}$, $\mathbb{R}_{\langle 2\rangle}$, ..., $\mathbb{R}_{\langle n-1\rangle}$ are equivalent.

Proof: Suppose that $\mathbb{R}_{\langle s\rangle}$ does not satisfy transitivity $(s<n)$, then there are at least one $r_{ij}$ $(1\leq i \neq j \leq s)$ such that

$$\bigvee_{l=1}^{s}(r_{il}\wedge r_{jl})>r_{ij}$$

i.e. $$\bigvee_{l=1}^{n}(r_{il}\wedge r_{jl})\geqslant\bigvee_{l=1}^{s}(r_{il}\wedge r_{jl})>r_{ij} \qquad (4)$$

(4) shows that $\mathbb{R}$ is not equivalent.

Proposition 2.2 If a fuzzy matrix $\mathbb{R}$ is equivalent, then we have

$$r_{ij}=r_{ik}\wedge r_{kj}$$

for any $i,j=1,2,\dots,n$, $i\neq j$, where $r_{ik}$ is the equivalence level of $\mathbb{R}$.

Proof: Since $\mathbb{R}$ is equivalent, then we have

$$r_{ij}=\bigvee_{l=1}^{n}(r_{il}\wedge r_{lj})=\bigvee_{l\neq k}(r_{il}\wedge r_{lj})\vee(r_{ik}\wedge r_{kj})\geqslant r_{ik}\wedge r_{kj}$$

i.e. $$r_{ij}\geqslant r_{ik}\wedge r_{kj}. \qquad (5)$$

And $r_{kj}=\bigvee_{l=1}^{n}(r_{kl}\wedge r_{lj})=\bigvee_{l\neq i}(r_{kl}\wedge r_{lj})\vee(r_{ki}\wedge r_{ij})\geqslant r_{ki}\wedge r_{ij}$

i.e. $r_{kj}\geqslant r_{ki}\wedge r_{ij}$. Because $r_{ki}=r_{ik}$ is the equivalence level of $\mathbb{R}$, then we have $r_{kj}>r_{ij}$, hence

$$r_{ij}\leqslant r_{ik}\wedge r_{kj} \qquad (6)$$

(5) and (6) imply $r_{ij}=r_{ik}\wedge r_{kj}$.

Proposition 2.3 Let nxn main submatrix $\mathbb{R}_{<n>}$ of $(n+1)\times(n+1)$ fuzzy matrix $\mathbb{R}$ be equivalent, and $\mathbb{R}$ be reflexive and symmetric. If entries $r_{ij}$ of $\mathbb{R}$ satisfy the following equalities

$$r_{ij}=r_{ik}\wedge r_{kj},\quad i,j=1,2,\dots,n+1,\ i\neq j,\ i\neq k,$$

then $\mathbb{R}$ is also equivalent, where $r_{ik}$ is the equivalence level of $\mathbb{R}$.

Proof: By way of a contradiction suppose that $(n+1)\times(n+1)$ matrix $\mathbb{R}$ is not equivalent, i.e. there exist $i,j$ such that

$$r_{ij}\neq\bigvee_{l=1}^{n+1}(r_{il}\wedge r_{lj})=\bigvee_{l=1}^{n}(r_{il}\wedge r_{lj})\vee(r_{i,n+1}\wedge r_{n+1,j}).$$

Since $r_{ij}=\bigvee_{l=1}^{n}(r_{il}\wedge r_{lj})$, hence

$$r_{ij}<r_{i,n+1},\quad r_{ij}<r_{n+1,j}=r_{j,n+1}. \qquad (7)$$

We consider the following two cases respectively.

1. There are at least an equivalence level of the (n+1)th columm of $\mathbb{R}$ at $r_{i,n+1}$ or $r_{j,n+1}$, where $i\neq n+1$, $j\neq n+1$.

Suppose that $r_{i,n+1}$ ($i\neq n+1$) is an equivalence level of the (n+1)th columm of $\mathbb{R}$. Since $\mathbb{R}$ is symmetric, then $r_{i,n+1}=r_{n+1,i}$, $r_{j,n+1}=r_{n+1,j}$. By the assumptions, we have $r_{n+1,j}=r_{n+1,i}\wedge r_{ij}$, i.e. $r_{j,n+1}=r_{i,n+1}$

$\wedge r_{ij}$ , which contradicts (7).

2. Both $r_{i,n+1}$ and $r_{j,n+1}$ are not equivalence level of the (n+1)th columm of $\mathbb{R}$. where $i \neq n+1$, $j \neq n+1$.

Suppose that $r_{t,n+1}$ is an equivalence level of $\mathbb{R}$, $t \neq i$, $t \neq j$ . By symmetry and the assumptions, we have

$$r_{i,n+1} = r_{t,n+1} \wedge r_{it} ,$$

$$r_{j,n+1} = r_{t,n+1} \wedge r_{jt} ,$$

because $r_{t,n+1} > r_{i,n+1}$ , $r_{t,n+1} > r_{j,n+1}$ , thus $r_{it} = r_{i,n+1}$ , $r_{jt} = r_{j,n+1}$, which contradicts (7).

This completes our proof.

Proposition 2.4 A necessary and sufficient condition that a fuzzy similar matrix $\mathbb{R}$ is equivalent is that for any $i, j = 1, 2, \ldots, s$, $i \neq j$ and $s = 1, 2, \ldots, n$, each order main submatrices $\mathbb{R}_{\langle s \rangle}$ satisfy the following

$$r_{ij} = r_{ik} \wedge r_{kj}$$

where $r_{ik}$ is the equivalence level of $\mathbb{R}_{\langle s \rangle}$.

Proof: The proof is obvious.

Proposition 2.5 The entries of a fuzzy matrix $\mathbb{R}^k$ are denoted by $r_{ij}^{(k)}$, where $\mathbb{R}^2 = \mathbb{R} \circ \mathbb{R}$, $\mathbb{R}^k = \mathbb{R}^{k-1} \circ \mathbb{R}, k = 2, 3, \ldots$. If $r_{ij}$ is an equivalence level of $\mathbb{R}$, then for any positive integer k, $r_{ij}$ is still an equivalence level of $\mathbb{R}^k$, i.e. $r_{ij}^{(k)} = r_{ij}$ .

Proof: Since $r_{il} \leqslant r_{ij}$ ( $l \neq i$ , $l \neq j$ ), by (1), then we have $r_{il}^{(k)} \leqslant r_{ij}$ . Because $r_{ij}^{(2)} = \bigvee_{l=1}^{n} (r_{il} \wedge r_{lj}) = \bigvee_{l \neq i} (r_{il} \wedge r_{lj}) \vee r_{ij} \geqslant r_{ij}$ , thus $r_{ij}^{(2)} = r_{ij}$ , and $r_{ij}$ is still an equivalence level of $\mathbb{R}^2$. By the induction method, we can easily obtain the result.

Definition 2.3 Let $r_{ij}$ be an equivalence level of the fuzzy similar matrix $\mathbb{R}$, the pair $(i, j)$ is called an equivalence point of $\mathbb{R}$ (in fact, the pair $(i, j)$ is a point on Cartesia product space $X \times X$ when we regard $\mathbb{R}$ as a fuzzy subset on $X \times X$ ).

From Definition we easily see that if $(i, j)$ is an equivalence point, then $(j, i)$ is also an equivalence point, where $i \neq j$ .Generally , $(i, j)$ and $(j, i)$ are regarded as the same equivalence point.

Definition 2.4 Let $u_1 = (a_1, a_2)$, $u_2 = (b_1, b_2)$ be two equivalence points of $\mathbb{R}$. We call that $u_1$ and $u_2$ are in same fuzzy equivalence chain iff $\exists\, 1 \leqslant i, j \leqslant 2$ such that $a_i = b_j$ .

We call that equivalence point $(i,k_1),(k_1,k_2),\ldots,(k_{m-1},k_m)$, $(k_m,j)$ form a fuzzy equivalence chain, denoted by $e=\{i,k_1,\ldots,k_m,j\}$.

Definition 2.5 Suppose all equivalence points of $n\times n$ fuzzy matrix $\mathbb{R}$ form $m$ equivalence chains $e_1,e_2,\ldots,e_m$ whose union

$$\bigcup_{i=1}^{n} e = \{1,2,\ldots,n\} .$$

Let $\mathbb{E}=\{e_1,e_2,\ldots,e_m\}$, $\mathbb{R}^{\langle 2\rangle}=[r_{ij}^{\langle 2\rangle}]$ be a fuzzy similar relation matrix on $\mathbb{E}\times\mathbb{E}$, where $1\leqslant i,j\leqslant m$, $m\leqslant[\frac{n}{2}]$, $r_{ij}^{\langle 2\rangle}$ is a measure of similarity between two equivalence chains $e_i=\{i_1,i_2,\ldots,i_{k_1}\}$ and $e_j=\{j_1,j_2,\ldots,j_{k_2}\}$, we take that

$$r_{ij}^{\langle 2\rangle}=\bigvee_{\substack{u=1,\ldots,k_1\\ v=1,\ldots,k_2}} r_{i_u,j_v} , \quad r_{kk}^{\langle 2\rangle}=1 , \quad (k=1,2,\ldots,m).$$

where $r_{i_u,j_v}$ are the entries of the matrix $\mathbb{R}$. The equivalence points of $\mathbb{R}$ are called the second order equivalence points of $\mathbb{R}$, denoted by $(i,j)_2$.

Similarly, the fuzzy similar matrix $\mathbb{R}^{\langle N\rangle}$ can be constituted by the equivalence chains of $\mathbb{R}^{\langle N-1\rangle}$. The equivalence points of $\mathbb{R}^{\langle N\rangle}$ are called the N-th order equivalence points, $N=1,2,\ldots,$ denoted by $(i,j)_N$.

Proposition 2.6 If all equivalence points of $n\times n$ fuzzy similar matrix $\mathbb{R}$ form an equivalence chain, then each entries of the transitive closure $\mathbb{R}^*=[r^*_{ij}]$ for fuzzy similar matrix $\mathbb{R}$ are all equivalence levels of $\mathbb{R}$ except $r^*_{ii}=1$, $i=1,2,\ldots,n$.

Proof: We only show that $r^*_{ij}$ is equal to one of the equivalence levels of $\mathbb{R}$, for any $i,j=1,2,\ldots,n$, $i\neq j$.

Let $r_{i,k_1}, r_{k_1,k_2},\ldots,r_{k_l,j}$ be the equivalence levels relatively to equivalence points $(i,k_1),(k_1,k_2),\ldots,(k_l,j)$ of $\mathbb{R}$. By Proposition 2.5, we have $r^*_{i,k_1}=r_{i,k_1}$, $r^*_{k_1,k_2}=r_{k_1,k_2},\ldots,r^*_{k_l,j}=r_{k_l,j}$, and $(i,k_1)_*,(k_1,k_2)_*,\ldots,(k_l,j)_*$ are the equivalence points of $\mathbb{R}^*$. For any $i,j=1,2,\ldots,n$, $i\neq j$, by Proposition 2.2 we have the following

$$\begin{aligned} r^* &= r_{ik_1}\wedge r_{k_1,j} \\ &= r_{ik_1}\wedge r_{k_1,k_2}\wedge r_{k_2,j} \\ &\cdots\cdots\cdots\cdots \\ &= r_{ik_1}\wedge r_{k_1,k_2}\wedge\cdots\wedge r_{k_{l-1},k_l}\wedge r_{k_l,j} \end{aligned}$$

which means that $r^*_{ij}$ is equal to one of the equivalence levels

$r_{ik_1},\ldots,r_{k_l,j}$ of $\mathbb{R}$.

Proposition 2.7 Let $\mathbb{R}^* = [r^*_{ij}]_{n\times n}$ be the transitive closure of similar matrix $\mathbb{R} = [r_{ij}]_{n\times n}$. For any $i,j=1,2,\ldots,n$, $i \neq j$, there are at least a k such that $r^*_{ij} = r^{<k>}_{uv}$, where $r^{<k>}_{uv}$ is a k-th order equivalence level of $\mathbb{R}$.

Proof: First, we discuss a simple case. Let all first order equivalence points of nxn fuzzy similar matrix $\mathbb{R}$ form two chains $e_1 = \{1,2,\ldots,m\}$, $e_2 = \{m+1,m+2,\ldots,n\}$ ( If we obtain $e'_1 = \{i_1, i_2, \ldots, i_m$, $e'_2 = \{i_{m+1}, i_{m+2}, \ldots, i_n\}$, we can reorder the patterns of $X$ such that $i_k \to k$). We denote that

$U_{1,2} = \{ r_{ij} \mid r_{ij}$ is the entries of R, $i=1,2,\ldots,m$, $j=m+1, m+2,\ldots,n \}$,

$\mathbb{M} = \{ r_{ij} \mid (i,j)$ is first order equivalence points, $i,j = 1,2,\ldots,n \}$,

$\mathbb{R}_{<m>} = [r_{ij}]_{m\times m}$, $i,j=1,2,\ldots,m$.

$\mathbb{R}'_{<n-m>} = [r_{ij}]_{(n-m)\times(n-m)}$, $i,j=m+1,m+2,\ldots,n$.

i.e. $\mathbb{R}_{<m>}$ and $\mathbb{R}'_{<n-m>}$ are submatrices of $\mathbb{R} = [r_{ij}]_{n\times n}$.

1. When $r_{ij} \bar{\in} \mathbb{M}$ and $r_{ij} \bar{\in} U_{1,2}$, we can see that $r_{ij}$ is an entry of $\mathbb{R}_{<m>}$ or $\mathbb{R}'_{<n-m>}$ by the symmetry. Because $\mathbb{R}_{<m>}$ and $\mathbb{R}'_{<n-m>}$ satisfy the conditions of Proposition 2.6, hence $r^*_{ij} \in \mathbb{M}$.

2. Let $r_{ij}$ be a maximal entry on $U_{1,2}$. For any $l$, it is impossible that $r_{il}$ and $r_{lj}$ in the expression $r^{(2)}_{ij} = \bigvee_{l=1}^{n} ( r_{il} \wedge r_{lj} )$ are all elements of $\mathbb{M}$. And there exists an element in $\{r_{il}, r_{lj}\}$ for each $l=1,2,\ldots,n$ is the element of $U_{1,2}$. Hence we can easily see that $r^{(2)}_{ij} = r_{ij}$, and $r^{(k)}_{ij} = r_{ij}$ for any k.

Let $r_{ik} \in \mathbb{M}$, $r_{ij} \in U_{1,2}$, By Propositions 2.2 and 2.5, we have $r_{ij} = r_{kj}$, because $r_{ij} < r_{ik}$, $j=m+1,m+2,\ldots,n$, hence all row vectors of the submatrix $\mathbb{R}_U = [\alpha_{ij}]$ $(\alpha_{ij} \in U_{1,2})$ of $\mathbb{R}$ are the same. By the sym- symmetry, all columms are the same. This shows that the entries of $\mathbb{R}_U$ are all the same, they are just the equivalence levels of second order equivalence points of $\mathbb{R}$.

Second, we discuss the generally case. Let all first order equivalence points of nxn matrix $\mathbb{R}$ form k chains $e_1 = \{1,2,\ldots, m_1\}$, $e_2 = \{m+1, m+2,\ldots,m_2\}$, $\ldots$, $e_k = \{m_{k-1}+1, m_{k-1}+2,\ldots,m_k\}$, where

$m_1+m_2+\ldots+m_k=n$. We denote that

$$\mathbb{U}_{l,t}=\{r_{ij}\mid r_{ij} \text{ is the entries of } \mathbb{R}, i\in e_l, j\in e_t\}.$$

1'. If $r_{ij}\bar{\in}\mathbb{M}$ and $r_{ij}\bar{\in}\mathbb{U}_{l,t}$, we have $r^*_{ij}\in\mathbb{M}$ similar to 1.

2'. For any $l,t=1,2,\ldots,k$, $l\neq t$, the elements of $\mathbb{U}_{l,t}$ take the same value similar to 2.

3'. Let $r_{ij}\in\mathbb{U}_{lt}$ and for any $r_{ik}\in\mathbb{U}_{u,v}$ we have $r_{ij}\geqslant r_{ik}$ . because for any $l$ it is impossible that $r_{il}$ and $r_{lj}$ in the expression $r^{(2)}_{ij} = \bigvee_{l=1}^{n}(r_{il}\wedge r_{lj})$ are all elements of $\mathbb{M}$, hence we have $r^{(2)}_{ij}=r_{ij}$ . By analogies, $r^{(k)}_{ij}=r_{ij}$ for any positive integer k, i.e. $r^*_{ij}=r_{ij}$ .This shows that the equivalence levels of second order equivalence points of $\mathbb{R}$ are the corresponding entries of $\mathbb{R}^*$.

By analogies, for any k the equivalence levels of k-th order equivalence points of $\mathbb{R}$ are the corresponding entries of $\mathbb{R}^*$.

4'. Obviously, by the methods in Definition 2.5, we can form a similar matrix $\mathbb{R}^{\langle k\rangle}$ by nxn matrix $\mathbb{R}$ finally, and the equivalence points of $\mathbb{R}^{\langle k\rangle}$ form a chain, hence each entries of $\mathbb{R}^*$ are all equivalence levels of $\mathbb{R}$, by Proposition 2.6.

This completes our proof.

## 3. AN ALGORITHM OF FUZZY CLUSTERING

On the basis of the fuzzy similar matrix $\mathbb{R}$ given, by the Propositions 2.6 and 2.7, the procedure for direct clustering may be given as the following

I . Write all first order equivalence points $(i,j)$ and the levels $r_{ij}$ of $\mathbb{R}$, denoted by $r_{ij}/(i,j)_1$ .

II . Write the equivalence chains of $\mathbb{R}$. If all equivalence points of $\mathbb{R}$ form a chain, then turn to IV .

III . Write the maximal elements of set $\mathbb{U}_{i,j}$ of intersection points of each equivalence chains. Make matrix $\mathbb{R}^{\langle 2\rangle}$. repeat the procedure I , II , until the equivalence points of $\mathbb{R}^{\langle k\rangle}$ form a chain.

IV. Because a k-th order equivalence level $r^{\langle k\rangle}_{ij}$ is a coefficient of similarty and $r^{\langle k\rangle}_{ij}$ is an entry of $\mathbb{R}$ , then there exist fixed $u, v$ such that $r^{\langle k\rangle}_{ij}=r_{uv}$ , where $r_{uv}$ is the entry of $\mathbb{R}$. Hence, we may write $r^{\langle k\rangle}_{ij}/(i,j)_k$ as $r^{\langle k\rangle}_{ij}/(u,v)$ or $r_{uv}/(u,v)$, where $r_{uv}/(u,v)$ express the maximal possibility degree of which the patterns $u$ and $v$ belong to the same class according to the equivalence relation $\mathbb{R}^*$.

Finally, we make the clustering graph by $\{ r_{uv} / (u,v) \mid r_{uv}$ are the equivalence levels of $\mathbb{R}$, $1 \leqslant u, v \leqslant n. \}$ .

Example:

Let

$$\mathbb{R} = \begin{bmatrix} 1 & 0.9 & 0.7 & 0.6 & 0.4 & 0.3 \\ 0.9 & 1 & 0.4 & 0.2 & 0.5 & 0.6 \\ 0.7 & 0.4 & 1 & 0.8 & 0.1 & 0.5 \\ 0.6 & 0.2 & 0.8 & 1 & 0.5 & 0.3 \\ 0.4 & 0.5 & 0.1 & 0.5 & 1 & 0.95 \\ 0.3 & 0.6 & 0.5 & 0.3 & 0.95 & 1 \end{bmatrix}$$

Obviously, $\mathbb{R}$ is a similar matrix, not an equivalence matrix.

The first order equivalence points and levels are

$0.9/(1,2)$, $0.8/(3,4)$, $0.95/(5,6)$

respectively, the equivalence chains $e_1 = \{1,2\}$, $e_2 = \{3,4\}$, $e_3 = \{5,6\}$.

Make the matrix $\mathbb{R}^{\langle 2 \rangle}$.

$$r_{12}^{\langle 2 \rangle} = \bigvee_{\substack{i=1,2 \\ j=3,4}} r_{ij} = r_{13} = 0.7 , \quad r_{13}^{\langle 2 \rangle} = \bigvee_{\substack{i=1,2 \\ j=5,6}} r_{ij} = r_{26} = 0.6 , \quad r_{23}^{\langle 2 \rangle} = \bigvee_{\substack{i=3,4 \\ j=5,6}} r_{ij}$$

$$= r_{54} = 0.5 , \quad r_{kk}^{\langle 2 \rangle} = 1 , \; k=1,2,3.$$

hence

$$\mathbb{R}^{\langle 2 \rangle} = \begin{bmatrix} 1 & 0.7 & 0.6 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.5 & 1 \end{bmatrix}$$

The second order equivalence points and levels are $0.7/(1,2)_2$, $0.6/(3,1)_2$, they form a chain $e = \{1,2,3\}$. Hence, whole entries of $\mathbb{R}^*$ are $0.9/(1,2)$ , $0.8/(3,4)$ , $0.95/(5,6)$ , $0.7/(1,2)_2 = 0.7/(1,3)$ , $0.6/(3,1)_2 = 0.6/(2,6)$ , which are illustrated in Fig.1.

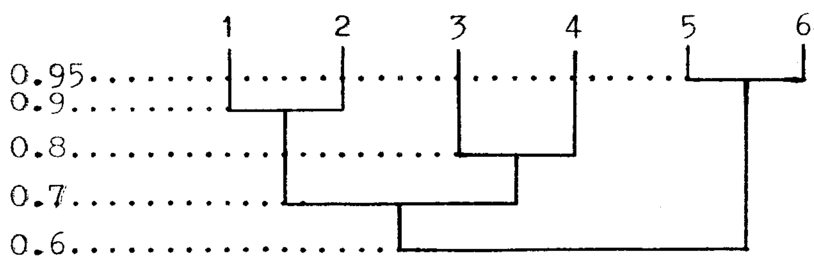


Fig 1. Hierarchical clustering graph.

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