ON MODEL OF ECONOMIC DYNAMICS IN A FUZZY ENVIRONMENT

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In this paper a concept of model of economic dynamics in a fuzzy environment is considered. The fuzzy environment is meant as fuzzy constraints imposed on subsequent states. Next the optimality of technological trajectories of this model is introduced and investigation.

1. Introduction.

In this paper a concept of model of economic dynamics in a fuzzy environment is considered. The fuzzy environment is meant as fuzzy constraints imposed on subsequent states. The fuzzy constraints and consequently the fuzzy goal, the fuzzy decision are in this case defined as fuzzy sets in the space of alternatives. The maximizing decision to be found is an trajectory which maximizes the membership function of the fuzzy goal.

In the third part of this paper the basic definitions and properties of multifunctions is presented. Whole multifunctions theory which is necessary for our considerations is presented in the monography of Makarov and Rubinov (1973).

In the next two parts the model of economic dynamics in a fuzzy environment is defined and the optimality of technological trajectories of this model is introduced and invwstigated.

The fuzzy sets theory which we use in this paper was initiated by Zadeh (1965). In our paper we use the following notation of fuzzy set: the fuzzy set A in the space X is defined as a function $\mu_A: X \to \langle 0,1 \rangle$. The function μ_A we will calle the membership function of A.

2. Notations.

Below are listed the principal mathematical notations used in this paper.

μ_A - the membership function of A ,

χ_B - the characteristic function of the set B ,

$\cap$ - set theoretic intersection,

$\subset$, $\supset$ - set theoretic inclusion "is contained" and "contains" respectively,

$\langle 0,1 \rangle$ - closed unit interval,

$a \wedge b = \min(a,b)$.

3. Basic definitions and properties of multifunctions.

Let A, B and C denote arbitrary but for our further considerations fixed sets. Next P(A), P(B) and P(C) denote respectively the families of all non-void subsets of A, B and C .
Let us consider a multifunction a from a set A into P(B) .

For any set $A^{\bullet} \in P(A)$ we set

$$a(A^{\bullet}) = \bigcup_{x \in A^{\bullet}} a(x) .$$

A converse multifunction, $a^{\leftarrow}$ say, to a multifunction $a: A \to P(B)$ is a multifunction such that

$$a^{\leftarrow}(y) = \{x \in A: y \in a(x)\} , \qquad (y \in a(A)) .$$

A composite, $b \circ a: A \to P(C)$ say, of two multifunctions $a: A \to P(B)$ and $b: B \to P(C)$ is a multifunction such that $\forall x \in A$

$$(b \circ a)(x) = b(a(x)) .$$

By the graph of a multifunction $a: A \to P(B)$ is understood the set

$$W_a = \{(x,y) \in A \times B: x \in A, y \in a(x)\} .$$

A multifunction $a: A \to P(B)$ is called closed iff its graph W_a is a closed set.

Now, let us consider a multifunction $a: K^{\bullet} \to P(K^{\bullet\bullet})$, where $K^{\bullet}$ and $K^{\bullet\bullet}$ denote convex and closed cones in the linear spaces $X^{\bullet}$ and $X^{\bullet\bullet}$ respectively.

A multifunction $a: K^{\bullet} \to P(K^{\bullet\bullet})$ is called superlinear iff it is:

(i) closed,

(ii) $a(0) = \{0\}$,

(iii) $a(\lambda x) = \lambda \cdot a(x) , \qquad \lambda > 0, x \in K^{\bullet}$,

(iv) $a(x_1 + x_2) \supset a(x_1) + a(x_2) , \qquad x_1, x_2 \in K^{\bullet}$,

(v) $a(K^{\bullet}) \cap (\text{int } K^{\bullet\bullet}) \neq \emptyset$.

4. Model of economic dynamics in a fuzzy environment.

Let E be a crisp subset of R_+ with at least two different elements. It is assumed that $0 \in E$. Elements of E will be called time moments and the element 0 - initial time moment.

Let $\tilde{E} = \{(t,\tau) \in E \times E: t < \tau\}$.

Let us consider the model of economic dynamics (compare e.g. [2])

$\tilde{\mathcal{M}} = \{E, (R^{n_t})_{t \in E}, (K_t)_{t \in E}, (a_{t\tau})_{(t,\tau) \in \tilde{E}}\}$,

where

- R^{n_t} denotes n_t - dimensional Euclidean space,
- K_t - convex and closed cone in R^{n_t},
- $a_{t\tau}$ - superlinear multifunction from K_t into $P(K_\tau)$.

It is assumed that the class $(a_{t\tau})_{(t,\tau) \in \tilde{E}}$ has the following property:

if $t, \tau, \Theta \in E$ and $t < \Theta < \tau$ then $a_{t\tau} = a_{\Theta\tau} \circ a_{t\Theta}$.

The cones K_t and K_τ are consider as the sets of all goods which are consumed or produced by the technology $a_{t\tau}$. So, for any good $x \in K_t$, $a_{t\tau}(x)$ denotes the set of all goods which are obtained by the technology $a_{t\tau}$ from the raw x.

Now, let us assumed that an information about all goods at given time moment is given. For example, let us assume that an information about consumer's preferences of the goods is given. These informations we can describe using the notion of fuzzy set. Namely, let C_t denotes the fuzzy set in R^{n_t}. This fuzzy set plays the role of a fuzzy constraint in the consumer's economic calculation at the time moment t. The number $\mu_{C_t}(x)$ expresses the degree of satisfaction with the fuzzy constraint, i.e. the degree of consumer's preferences of the good x.

Now, we can present the definition of a model of economic dynamics in a fuzzy environment.

Definition 4.1. A model of economic dynamics in a fuzzy environment is an object

$$\mathcal{M} = \{E \,,\ (R^{n_t})_{t\in E} \,,\ (K_t)_{t\in E} \,,\ (C_t)_{t\in E} \,,\ (a_{t\tau})_{(t,\tau)\in \tilde{E}}\} \,.$$

It is assumed that for any $t \in T$, C_t is convex (Zadeh 1965) and closed (Kloeden 1982) fuzzy set in R^{n_t} .

Let us note, that if for any $t \in E$ and for any $x \in R^{n_t}$ $\mu_{C_t}(x) = 1$ then we will obtain the model of economic dynamics, [2].

Definition 4.2. A technological trajectory of $\mathcal{M}$ is a family $Tr = (x_t)_{t\in E}$ such that

- $x_t \in K_t$, $(t \in E)$,
- $x_\tau \in a_{t\tau}(x_t)$, $((t,\tau) \in \tilde{E})$,
- $\mu_{C_t}(x_t) > 0$, $(t \in E)$.

In this case x_t is called the state of the trajectory Tr at the time t, x_0 is the initial state of Tr . It is said that the trajectory Tr goes out x if $x = x_0$ and passes through x at the time t if the state of Tr at the time t is x .

Now, let us consider the multifunction $a_{t\tau}$ and the fuzzy set C_t such that

- if $(\chi_{a_{t\tau}(x_t)} \wedge \mu_{C_\tau})(x_\tau) > 0$ then $\forall\ \theta \in (t,\tau)$

$$\chi_{a_{t\theta}(x_t) \cap a^{-}_{\theta\tau}(x_\tau)} \wedge \mu_{C_\theta} \neq 0 \,.$$

If the above property holds then we will say that the FC property holds.

Theorem 4.2.1. (of the existence of technological trajectory). Let $0, \overline{T} \in E$, $0 < \overline{T}$, $y_0 \in K_0$, $y \in a_{0\overline{T}}(y_0)$, $\mu_{C_0}(y_0) > 0$, $\mu_{C_{\overline{T}}}(y) > 0$ and let the FC property holds. Then there exists a technological trajectory Tr of $\mathcal{M}$ going out y_0 and passing through y at the time $\overline{T}$.

Proof. For every $t \in E$ let us take an element $w_t \in R^{n_t} \setminus K_t$ and let us denote $L = \prod_{t \in E} K_t$ (a Cartesian product) . Next let us choose a subset $M \subset L$ such elements $Tr = (x_t)_{t \in E}$ that there exists a subset $E_{Tr} \subset E$ such that:

- $0, \bar{t} \in E_{Tr}$,
- if $t', t'' \in E_{Tr}$ and $t' < t''$ then $x_{t''} \in a_{t't''}(x_{t'})$ and $\mu_{C_{t''}}(x_{t''}) > 0$,
- $x_0 = y_0$, $x_{\bar{t}} = y$,
- if $t \in E \setminus E_{Tr}$ then $x_t = w_t$.

By the way, let us mention that $M \neq \emptyset$. Indeed, the element $Tr = (x_t)_{t \in E}$ such that $x_0 = y_0$, $x_{\bar{t}} = y$, $x_t = w_t$ if $t \in E$ and $t \neq 0, \bar{t}$ belongs to M and $E_{Tr} = \{0, \bar{t}\}$. In M one can introduce a partial ordering $\succcurlyeq$ as follows:

$$Tr^1 \succcurlyeq Tr^2 , \quad Tr^i = (x_t^i)_{t \in E} \subset M , \quad i=1,2$$

iff

- $E_{Tr^1} \supset E_{Tr^2}$,
- $x_t^1 = x_t^2, \quad \forall\, t \in E_{Tr^2}$.

Now, it will be checked that each chain in M is bounded from above. Really, let $(Tr^\alpha)_{\alpha \in A}$ denote a chain in M , $Tr^\alpha = (x_t^\alpha)_{t \in E}$, $\alpha \in A$. It is seen that the element $Tr = (x_t)_{t \in E}$ such that

- $x_t = x_t^\alpha$ if $t \in E_{Tr^\alpha}$,
- $x_t = w_t$ if $t \notin \bigcup_{\alpha \in A} E_{Tr^\alpha}$,

belongs to M and $E_{Tr} = \bigcup_{\alpha \in A} E_{Tr^\alpha}$. Moreover, for every $\alpha \in A$ $Tr \succcurlyeq Tr^\alpha$. So, the chain $(Tr^\alpha)_{\alpha \in A}$ is bounded from above, as it was anerted. By Zorn's Lemma the set M has a maximal element. It remains now to prove that for each maximal element $Tr = (x_t)_{t \in E}$ of M there holds $E_{Tr} = E$.

Indeed, let us suppose that there exists such maximal element $Tr \subset M$ that $E \setminus E_{Tr} \neq \emptyset$. So, there exists an element $\theta \in E \setminus E_{Tr}$.

Now, let us define

$F_1 = \{ t \in E_{Tr}: t < \theta \}$, $F_2 = \{ \tau \in E_{Tr}: \theta < \tau \}$, $F = F_1 \times F_2$

and for $(t,\tau) \in F$

$b_{t\tau} = \{ x_\theta : (\chi_{a_{t\theta}(x_t) \cap a^{\leftarrow}_{\theta\tau}(x_\tau)} \wedge \mu C_\theta)(x_\theta) > 0 \} \subset R^{n_\theta}$.

Let us note that the set $b_{t\tau}$ has the following properties:

(1) The set $b_{t\tau}$ is a non empty set.

Really, let $t < \theta < \tau$. Because x_t, $x_\tau \in Tr$ we observe that

$$(\chi_{a_{t\tau}(x_t)} \wedge \mu C_\tau)(x_\tau) > 0 .$$

Next, taking into account the FC property, we get

$$\chi_{a_{t\theta}(x_t) \cap a^{\leftarrow}_{\theta\tau}(x_\tau)} \wedge \mu C_\theta \neq 0 .$$

Therefore, there exists x_θ such that

$$(\chi_{a_{t\theta}(x_t) \cap a^{\leftarrow}_{\theta\tau}(x_\tau)} \wedge \mu C_\theta)(x_\theta) > 0 , \quad \text{i.e.} \quad b_{t\tau} \neq \emptyset .$$

(2) The set $b_{t\tau}$ is a compact set.

The multifunctions $a_{t\theta}$ and $a_{\theta\tau}$ are superlinear, so they are closed multifunctions. Moreover the fuzzy set C_θ is closed. So, the set $b_{t\tau}$ is an intersection of there sets, a closed one, a compact one and a closed one. Therefore, $b_{t\tau}$ is a compact set.

(3) if (t^1,τ) , $(t^2,\tau) \in F$, $t^1 < t^2$ then $b_{t^1\tau} \supset b_{t^2\tau}$.

For $t^1 < t^2 < \tau$ we have

$$\chi_{a_{t^1\theta}(x_{t^1})} \wedge \mu C_\theta = \chi_{a_{t^2\theta} \circ a_{t^1 t^2}(x_{t^1})} \wedge \mu C_\theta .$$

So,

$$a_{t^2\theta}(x_{t^2}) \subset a_{t^1\theta}(x_{t^1})$$

and therefore

$$b_{t^1\tau} \supset b_{t^2\tau} \; .$$

(4) If (t, τ^1) , $(t, \tau^2) \in F$, $\tau^1 < \tau^2$ then $b_{t\tau^1} \subset b_{t\tau^2}$.

From (1), (3) and (4) it follows that the family $(b_{t\tau})_{(t,\tau) \in F}$ is cantered.

Let $(t^i, \tau^i) \in F$, $i = 1(1)n$, $\tau = \min_i \tau^i$, $t = \max_i t^i$. It seen that $(t,\tau) \in F$. Moreover, if (t,τ) , (t^i,τ) , $(t^i, \tau^i) \in F$, $(i=1(1)n)$, then $b_{t\tau} \subset b_{t^i\tau^i}$. So, $b_{t\tau} \subset \bigcap_{i=1}^{n} b_{t^i\tau^i}$. But $b_{t\tau} \neq \emptyset$, therefore $\bigcap_{i=1}^{n} b_{t^i\tau^i} \neq \emptyset$. Compactness of $b_{t\tau}$ and centerness of the family $(b_{t\tau})_{(t,\tau) \in F}$ yields $\bigcap_{(t,\tau) \in F} b_{t\tau} \neq \emptyset$. Let an element $z_\Theta \in \bigcap_{(t,\tau) \in F} b_{t\tau}$ and let us take into account an element $\overline{Tr} = (\bar{x}_t)_{t \in E}$ such that

$$\bar{x}_t = \begin{cases} x_t & \text{if} \quad t \in E_{Tr} \, , \\ z_\Theta & \text{if} \quad t = \Theta \, , \\ w_t & \text{if} \quad t \in E \setminus (E_{Tr} \cup \{\Theta\}) \, . \end{cases}$$

For this element we have $E_{\overline{Tr}} = E_{Tr} \cup \{\Theta\}$. So, we get $\overline{Tr} \succ Tr$ and $\overline{Tr} \neq Tr$ in contradiction with supposed fact that Tr is the maximal element in M . This means - in wnother words - that there exists a technological trajectory of $\mathcal{M}$ with initial state y_0 and passing through y at the time moment $\bar{t}$.

5. Optimal trajectories in dynamics model.

Now, let us additionally assume that there exists $T \in E$ such that $\forall \; t \in T$, $t \leq T$.

Let

$$b_{OT}(x_O) = \{x_T : x_T \in a_{OT}(x_O),\ \mu_{C_T}(x_T) > 0 \text{ and } \forall\ t \in (0,T)$$
$$\chi_{a_{Ot}(x_O) \cap \overleftarrow{a}_{tT}(x_T)} \wedge \mu_{C_t} \neq 0\}.$$

The set $b_{OT}(x_O)$ is the set of all goods which are obtained by the technology a_{OT} from the raw x_O and which have the positive degree of consumers preferences.

Let G_T denotes a fuzzy set of R^{n_T} such that $\forall\ x \in R^{n_T}$

$$\mu_{G_T}(x) = \chi_{b_{OT}(x_O)}(x) \wedge \mu_{C_T}(x).$$

The fuzzy set G_T we will call the fuzzy goal.

In some cases preferences between alternatives may be described by a so called utility function. This function maps a given set of alternatives into a set of estimates of alternatives and a preference relation is specifies in this last set. This function therefore allows to compare alternatives with each other by their estimates. In the fuzzy case utility functions may have various forms. One of the form is the fuzzy set of alternatives (the fuzzy goal).

Definition 5.1. A technological trajectory Tr with the initial state x_O and terminal state x_T is called μ - optimal if

$$\mu_{G_T}(x_T) = \sup_{y \in K_T} \mu_{G_T}(y).$$

The μ - optimality means that this trajectory is optimal which the terminal state has the superlative degree of consumer's preferences.

Theorem 5.1. A technological trajectory Tr with the initial and terminal states x_O and x_T respectively is μ - optimal iff the separation degree of the fuzzy sets G_T and $\{x_T\}_{G_T}$ is equal to

$$1 - \sup_{x \in K_T} \mu_{G_T}(x),$$

where $\{x_T\}_{G_T}$ is the fuzzy subset of R^{n_T} such that $\forall y \in R^{n_T}$

$$\mu_{\{x_T\}_{G_T}}(y) = \begin{cases} \mu_{G_T}(x_T) & \text{if} \quad y = x_T , \\ 0 & \text{if} \quad y \neq x_T . \end{cases}$$

Proof. Let Tr denote a μ - optimal trajectory with initial and terminal states x_0 and x_T respectively. Because the fuzzy sets G_T and $\{x_T\}_{G_T}$ are bounded and convex so their separation degree is equal to

$$1 - \sup_{x \in K_T} (\mu_{G_T}(x) \wedge \mu_{\{x_T\}_{G_T}}(x)) = 1 - \mu_{G_T}(x_T) = 1 - \sup_{x \in K_T} \mu_{G_T}(x) .$$

Let now the separation degree of fuzzy sets G_T and $\{x_T\}_{G_T}$ will be equal to $1 - \sup_{x \in K_T} \mu_{G_T}(x)$. So, in accordance with The Separations Theorem for fuzzy sets we have

$$\sup_{x \in K_T} \mu_{G_T}(x) = \sup_{x \in K_T} (\mu_{G_T}(x) \wedge \mu_{\{x_T\}_{G_T}}(x)) = \mu_{G_T}(x_T) .$$

Therefore the trajectory Tr is μ - optimal.

An element x of a set A ($A \neq \emptyset, \{0\}$) is called the limiting point from above of A if $a \cdot x \notin A$ for $a > 1$.
For a normal covering of A we use the symbol **nA** (compare Makarov and Rubinov 1973).

Let K_T^{*} denotes a set of all linear functionals $f: K_T \to \langle 0, \infty)$.

Definition 5.2. A technological trajectory Tr with initial and terminal states x_0 and x_T respectively is called p-optimal if there exists a non-zero functional $p \in K_T^{*}$ such that

$$p(x_T) = \max_{x \in \text{supp } G_T} p(x) > 0 \qquad (*)$$

The functional p we may interpret as prices of all goods from K_T. Then p-optimality of technological trajectory means that this trajectory is optimal which the terminal state has the highest value with the prices p .

Theorem 5.2.1. A technological trajectory Tr with initial and terminal states x_0 and x_T respectively is p-optimal iff the element x_T is a limiting point from above of the set nsupp G_T .

Proof. Let Tr denote a p-optimal trajectory with initial and terminal states x_0 and x_T respectively. We are going to prove that x_T is a limiting point from above of nsupp G_T . In contrary, there exists $a > 1$ such that $a \cdot x_T \in$ nsupp G_T . Because of $p(x_T) > 0$ we get

$$p(x_T) = \max_{x \in \text{supp } G_T} p(x) = \max_{x \in \text{supp } G_T} p(x) \geq p(a \cdot x_T) = a\, p(x_T) > 0 ,$$

i.e. $1 \leq a$ in contradiction with $a > 1$.

Now, let us assume that x_T is a limiting point from above of the set nsupp G_T . Let S denote the aphere nsupp G_T – nsupp G_T and $\|\cdot\|_{\text{nsupp } G_T}$ – norm of Minkowski. It is known that this norm is monotonous and

$$\text{nsupp } G_T = \{ z: \ \|z\|_{\text{nsupp } G_T} \leq 1 \} .$$

Because x_T nsupp G_T is simulteneously a limiting point from above of the set nsupp G_T , there holds $\|x_T\|_{\text{nsupp } G_T} = 1$. Therefore, there exists a functional $p \in K_T^{*}$ such that $p(x_T) = \|x_T\|_{\text{nsupp } G_T} = 1$, $\|p\| = 1$.

So, it is proved that the trajectory Tr the condition (*) is fulfilled and thus the proof is finished.

References

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