# A MULTI-LAYER FUZZY B-D TYPE ALGEBRAIC STRUCTURE IN THE COMPLEX SYSTEM

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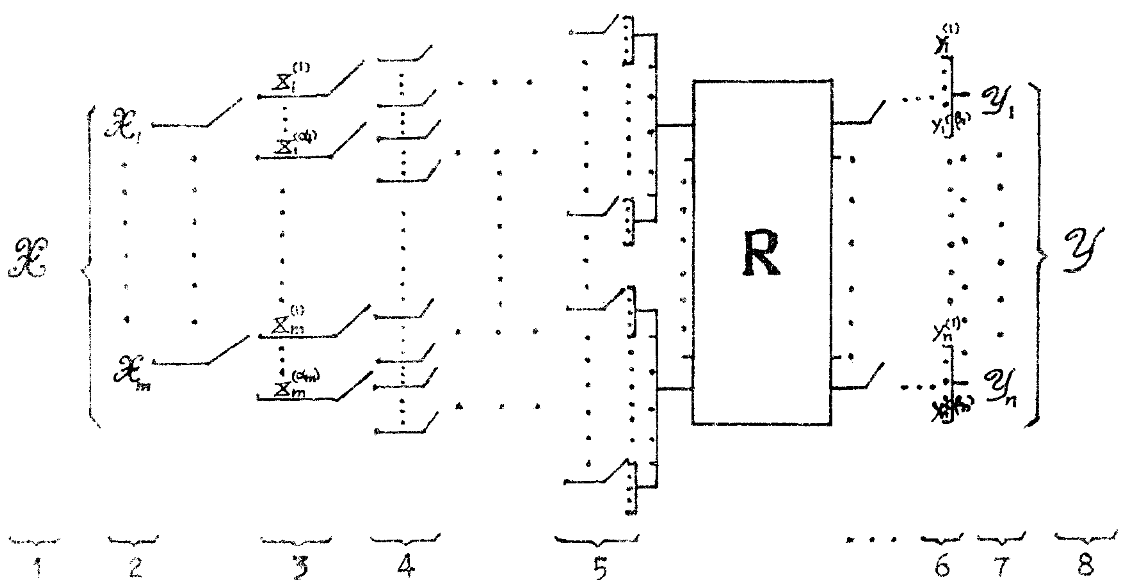
ABSTRACT

The purpose of this article is to seek for a method in the research of the fuzzy control decision in the complex system. That is to study the numerical relations among their variables throngh examining the logical relationships among the data. Therefore, a multi-layer fuzzy B-D type algebraic system is set up, and with the help of the language method the logical analysis for the complex system is realized.

## I. Mathematical Model Setting

The multi-layer logical structure of the complex system $\mathcal{P}$ is shown in the following imitative form:




Notes:

1. the state space of the input variables.
2. $\mathcal{X}_i$ is the language variables (or the true-false variables of the language).
3. $X_i^{(j)}$ is one layer fuzzy variable (input).
4. two layers fuzzy variable.
5. n layers fuzzy variable.
6. $y_i^{(j)}$ is the fuzzy variable (output).
7. $\mathcal{Y}_i$ is the language variables (or the true-false variables of the language).
8. the state space of the output variables.

In the above model, for the horizontal levels the main object of the study is the relation between the layers of the language variable $\mathscr{X}_i$ and the fuzzy variable $x_i^j$, or they could be considered as two typical layers; and the relation between the i layer and the i (i=1, ..., k-1) fuzzy variables is the reapperance and wideness of these two typical layers. For the vertical layers the main object of the study is the logical relations among the several internal fuzzy variables (called the B layer) of the same language variable $\mathscr{X}_i$ in the input state space, as well as the logical relations among the fuzzy variables (called the D layer) of the different species in the different language variables.

Practically the above modular form is set up according to the concept called the turning mapping.

**Definition1.** Let us call mapping $f: U \to W$ be the turning mapping, if U is the field theory of the basic variables, W is the word set reflecting the corresponding changing state of the given state space and $\forall u \in [a, b] \subseteq U$, there is a unique word in correspondence, that is, $f(u)=w$; vice versa.

For example, considering the test of the vinylon formaling as a complex system, the following table show how the above steps are carried out:

Table1. The state words (fuzzy variables)

| K \ T / S | $x_1^{(1)}$ | $x_1^{(2)}$ | $x_1^{(3)}$ | $x_1^{(4)}$ | $x_1^{(5)}$ | $x_1^{(6)}$ |
|---|---|---|---|---|---|---|
| $x_2^{(1)}$ | $y_1^{(1)}$ | $y_1^{(1)}$ | $y_1^{(2)}$ | $y_1^{(4)}$ | $y_1^{(4)}$ | $y_1^{(5)}$ |
| $x_2^{(2)}$ | $y_1^{(1)}$ | $y_1^{(2)}$ | $y_1^{(3)}$ | $y_1^{(4)}$ | $y_1^{(5)}$ | $y_1^{(5)}$ |
| $x_2^{(3)}$ | $y_1^{(1)}$ | $y_1^{(2)}$ | $y_1^{(3)}$ | $y_1^{(4)}$ | $y_1^{(4)}$ | $y_1^{(5)}$ |

T(minute) ———— reaction time.

S( g/l ) ———— density of formaldehyde.

K(gram molecule%) ——— degree of formaling.

$x_1^{(1)}$: very short, ...; $x_2^{(1)}$: not thick, ...;

$y_1^{(1)}$: low, ... the meanings of the words could be given in a corresponding consistent function.

## II. B-D type calculus system

(1) B-layer calculus:

The discussion is limited in $\mathscr{X}_i$ (the rest may be inferred by analogy), and every $x_i^j$ is explained in the meanings of the language variables. Assume $\mathscr{X}_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)})$, including three operators $\odot$,

$\oplus$, $\_^*$ and elements $0_i, 1_i$ $(i=1,\ldots,m)$ and so on. As every variable is in its definite occasion, take and only take one in the r-state and stipulate:

**Definition2.** Let $x_i^{(p)} \triangleq \int_{U_i} \mu_{x_i^{(p)}}(u)/u$ , $x_i^{(q)} \triangleq \int_{U_i} \mu_{x_i^{(q)}}(u)/u$ , then

$$x_i^{(p)} \odot x_i^{(q)} \triangleq \begin{cases} \int_{U_i} 0/u & (p\neq q) \\ \int_{U_i} [\mu_{x_i^{(p)}}(u) \wedge \mu_{x_i^{(q)}}(u)]/u & (p=q) \end{cases}$$

$$x_i^{(p)} \oplus x_i^{(q)} \triangleq \int_{U_i} [\mu_{x_i^{(p)}}(u) \vee \mu_{x_i^{(q)}}(u)]/u$$

$$\overline{x_i^{(p)}}^* \triangleq \int_{U_i} [\mu_{x_i^{(1)}}(u) \vee \cdots \vee \mu_{x_i^{(p-1)}}(u) \vee \mu_{x_i^{(p+1)}}(u) \vee \cdots \vee \mu_{x_i^{(r)}}(u)]/u$$

$$0_i \triangleq \int_{U_i} 0/u$$

$$1_i \triangleq \int_{U_i} [\mu_{x_i^{(1)}}(u) \vee \cdots \vee \mu_{x_i^{(r)}}(u)]/u$$

In it , $u \in U_i$ ($U_i$ is the field theory of $\mathscr{X}_i$), $\vee \triangleq$ Max , $\wedge \triangleq$ Min.

From the law of the equivalent power, the simple expressive formula of the above can be obtained.:

$$x_i^{(p)} \odot x_i^{(q)} \triangleq \begin{cases} 0_i & (p\neq q) \\ x_i^{(p)} & (p=q) \end{cases}$$

$$x_i^{(p)} \oplus x_i^{(q)} \triangleq \begin{cases} \int_{U_i} [\mu_{x_i^{(p)}}(u) \vee \mu_{x_i^{(q)}}(u)]/u & (p\neq q) \\ x_i^{(p)} & (p=q) \end{cases}$$

$$\overline{x_i^{(p)}}^* \triangleq x_i^{(1)} \oplus \ldots \oplus x_i^{(p-1)} \oplus x_i^{(p+1)} \oplus \ldots \oplus x_i^{(r)} = \bigoplus_{j\neq p} x_i^{(j)}$$

$$1_i \triangleq x_i^{(1)} \oplus \ldots \oplus x_i^{(r)} = \bigoplus_{j=1}^{r} x_i^{(j)}$$

Its clear that $0_i$ and $1_i$ are two special elements different from $x_i^{(j)}$ $(j=1,\ldots,r)$. They are stipulated as :

Table 2

| $\odot$ ( $\oplus$ ) | $0_i$ | $1_i$ |
|---|---|---|
| $0_i$ | $0_i(0_i)$ | $0_i(1_i)$ |
| $1_i$ | $0_i(1_i)$ | $1_i(1_i)$ |

Note: Here only the continuous model is discussed, as for the discrete model it is on the analogy of this.

(2) D-layer calculus:

In the calculus of the fuzzy variables in the different species, every $x_i^{(s)}, x_j^{(t)}, \ldots$ is explained in the meanings of the true-false variables of the Boolean language. It is provided that these three operators $\cdot, +, -$, are "and", "or" and "non", the logical meanings respectively. The fuzzy variables in different language variables are linked by these operators. The ontput variables are expressed by the specific logical relations among them.

It is ordered that $\nu(x_i^{(j)}) = \int_0^1 \mu_t/\nu_t$ or $\nu(x_i^{(j)}) = \sum_{t=1}^{n} \mu_t/\nu_t$ , in it, $\nu_t$ is the points in $[0,1]$, and $\mu_t$ is the classes of their qualifications in $\nu(x_i^{(j)})$.

Assume X in the language value of the Boolean language variable $\mathcal{G}$, T is that of the Boolean true and false language variable $\mathcal{Y}$, we shall have the homomorphic mapping $\nu$ from the $\mathcal{G}$ dictionary to the $\mathcal{Y}$ dictionary, so $\nu(x)=T$.

For $x_i^{(s)}\in\mathcal{X}_i$ and $x_j^{(t)}\in\mathcal{X}_j$ the following regulations are given:

**Definition 3.** Let $\nu(x_i^{(s)})\triangleq\int_0^1\alpha_i/\nu_i$ , $\nu(x_j^{(t)})\triangleq\int_0^1\beta_j/\omega_j$ , then

$$x_i^{(s)}\cdot x_j^{(t)}\triangleq\int_0^1(\alpha_i\wedge\beta_j)/(\nu_i\wedge\omega_j)$$

$$x_i^{(s)}+x_j^{(t)}\triangleq\int_0^1(\alpha_i\wedge\beta_j)/(\nu_i\vee\omega_j)\qquad \overline{x_i^{(s)}}\triangleq\int_0^1\alpha_i/1-\nu_i$$

Here $\overline{x_i^{(s)}}$ and $\overline{x_i^{(s)}}^{*}$ are two different expressions of the meanings of words. In order to tell one from another its better to call the former (for the whole system $\mathcal{F}$) the whole complement and the latter (for $\mathcal{X}$) the relative complement. In it

$\wedge\ \triangleq$ conjunction (be different from $\cap$)

$\vee\ \triangleq$ disjunction (be different from $\cup$)

$$\mathcal{U}_T(\nu)=\begin{cases}0 & 0\le\nu\le a\\ 2\left(\frac{\nu-a}{1-a}\right)^2 & a\le\nu\le\frac{a+1}{2}\\ 1-2\left(\frac{\nu-1}{1-a}\right)^2 & \frac{a+1}{2}\le\nu\le 1\end{cases}$$

In the D-layer calculus, there are two elements 0 and 1, they are two special true-false values (corresponding to the maximal and minimal elements of the lattice constructed by the fuzzy subset of the interval [0,1] ).

$$0\triangleq\int_0^1 0/\nu\qquad,\qquad 1\triangleq\int_0^1 1/\omega$$

For 0 and 1 the regulation is as follows:

Table 3

| $\cdot\,(+)$ | 0 | 1 |
|---|---|---|
| 0 | 0(0) | 0(1) |
| 1 | 0(1) | 1(1) |

(3) B-D type calculus system:

The B-D type calculus is mingled with the B-layer calculus and the D-layer calculus. For the calculus of the whole complex system, it is the development of the basic calculuses mentioned above.

Through the B-D type calculus not only the fuzzy formula F could be simplified, but also the degree of truth and accuracy of the given logical and numerical relations by F could be obtained.

For example, from the above vinylon formaling test tables it can be got that

$$y_1^{(5)}=x_1^{(5)}\cdot x_2^{(2)}+x_1^{(6)}\cdot x_2^{(1)}+x_1^{(6)}\cdot x_2^{(2)}+x_1^{(6)}\cdot x_2^{(3)}$$

$$\overset{(D)}{=}x_1^{(5)}\cdot x_2^{(2)}+x_1^{(6)}\cdot(x_2^{(1)}\oplus x_2^{(2)}\oplus x_2^{(3)})$$

$$\overset{(B)}{=}x_1^{(5)}\cdot x_2^{(2)}+x_1^{(6)}\cdot 1_2$$

$$\overset{(D)}{=}x_1^{(5)}\cdot x_2^{(2)}+x_1^{(6)}$$

( Here the calculus character on which the deformation is based will be worked out below ).

The consequent conclusion is that the reaction time should be considerably long ($x_1^{(5)}$) in order to let the degree of the formaling reach the highest point ($y_1^{(5)}$), or the reaction time is long ($x_1^{(5)}$) while the density of formaldehyde is medium ($x_2^{(2)}$). The language true and false value of $y_1^{(5)}$ expressive formula can be calculated, so the corresponding degree of faithfulness to the control decision provided by the above conclusion can be obtained.

Based on this, the field theory intervals can be divided more precisely and finely, B-D type calculus is carried out again, and moved into the deeper layers, then the "ideal state " is approached by and by and the optimum control is realized.

**III.** The axiomatized **structure** of the B-D type calculus theory

**Difinition 4. Let** $\underset{\sim}{\mathcal{B}}$ is the set of some fuzzy objects, it has two definite elements $\underset{\sim}{U}$ and $\underset{\sim}{\Phi}$ , and $\underset{\sim}{U} \neq \underset{\sim}{\Phi}$. If three algebraic calculuses $\uplus$, $\Cap$ and $-$ are defined, then for $\forall\ \underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C} \in \underset{\sim}{\mathcal{B}}$, the following 4 axioms are reasonable:

( Law of commutation ) $\underset{\sim}{A} \uplus \underset{\sim}{B} = \underset{\sim}{B} \uplus \underset{\sim}{A}$

$\underset{\sim}{A} \Cap \underset{\sim}{B} = \underset{\sim}{B} \Cap \underset{\sim}{A}$

(Law of distribution ) $\underset{\sim}{A} \uplus ( \underset{\sim}{B} \Cap \underset{\sim}{C} ) = ( \underset{\sim}{A} \uplus \underset{\sim}{B} ) \Cap ( \underset{\sim}{A} \uplus \underset{\sim}{C} )$

$\underset{\sim}{A} \Cap ( \underset{\sim}{B} \uplus \underset{\sim}{C} ) = ( \underset{\sim}{A} \Cap \underset{\sim}{B} ) \uplus ( \underset{\sim}{A} \Cap \underset{\sim}{C} )$

(Law of identity ) $\underset{\sim}{A} \uplus \underset{\sim}{\Phi} = \underset{\sim}{A}$

$\underset{\sim}{A} \Cap \underset{\sim}{U} = \underset{\sim}{A}$

(Law of complement**ary**) $\underset{\sim}{A} \uplus \overline{\underset{\sim}{A}} = \underset{\sim}{U}$

$\underset{\sim}{A} \Cap \overline{\underset{\sim}{A}} = \underset{\sim}{\Phi}$

Then $\langle \underset{\sim}{\mathcal{B}}, \uplus, \Cap, -, \underset{\sim}{U}, \underset{\sim}{\Phi} \rangle$ form a fuzzy **Boolean algebra .**

(1) B-layer calculus theory :

From the definition 2 the following properties can be induced :

(1a) $x_i^{(p)} \oplus x_i^{(p)} = x_i^{(p)}$

(1b) $x_i^{(p)} \odot x_i^{(p)} = x_i^{(p)}$

(2a) $x_i^{(p)} \oplus x_i^{(q)} = x_i^{(q)} \oplus x_i^{(p)}$

(2b) $x_i^{(p)} \odot x_i^{(q)} = x_i^{(q)} \odot x_i^{(p)}$

(3a) $x_i^{(p)} \oplus ( x_i^{(q)} \oplus x_i^{(r)} ) = ( x_i^{(p)} \oplus x_i^{(q)} ) \oplus x_i^{(r)}$

(4a) $x_i^{(p)} \odot ( x_i^{(q)} \oplus x_i^{(r)} ) = ( x_i^{(p)} \odot x_i^{(q)} ) \oplus ( x_i^{(p)} \odot x_i^{(r)} )$

(4b) $x_i^{(p)} \oplus ( x_i^{(q)} \odot x_i^{(r)} ) = ( x_i^{(p)} \oplus x_i^{(q)} ) \odot ( x_i^{(p)} \oplus x_i^{(r)} )$

(5a) $x_i^{(p)} \oplus 0_i = x_i^{(p)}$

(5b) $x_i^{(p)} \odot 1_i = x_i^{(p)}$

(6a) $x_i^{(p)} \oplus \overline{x_i^{(p)}}^{*} = 1_i$

(6b) $x_i^{(p)} \odot \overline{x_i^{(p)}}^{*} = 0_i$

From (2a),(2b),(4a),(4b),(5a),(5b),(6a),(6b), the following theorem can be obtained :

**Theorem 1.** $< \mathscr{X}_i , \odot , \oplus , -^* , o_i , 1_i >$ (i=1,...,n) form a fuzzy Boolean algebra.

Some other properties can be induced :

(7a) $x_i^{(p)} \oplus 1_i = 1_i$

(7b) $x_i^{(p)} \odot o_i = o_i$

(5b) $x_i^{(p)} \odot ( x_i^{(q)} \odot x_i^{(r)} ) = ( x_i^{(p)} \odot x_i^{(q)} ) \odot x_i^{(r)}$

(8) The uniqueness of $o_i$ and $1_i$ .

(9) The uniqueness of $\overline{x_i^{(p)}}^*$ complementation for $x_i^{(p)}$ .

(10) $\overline{o_i}^* = 1_i$ , $\overline{1_i}^* = O_i$

(11) $\overline{\overline{x_i^{(p)}}^*}^* = x_i^{(p)}$

(12a) $x_i^{(p)} \oplus ( x_i^{(p)} \odot x_i^{(q)} ) = x_i^{(p)}$

(12b) $x_i^{(p)} \odot ( x_i^{(p)} \oplus x_i^{(q)} ) = x_i^{(p)}$

(13a) $\overline{x_i^{(p)} \oplus x_i^{(q)}}^* = \overline{x_i^{(p)}}^* \odot \overline{x_i^{(q)}}^*$

(13b) $\overline{x_i^{(p)} \odot x_i^{(q)}}^* = \overline{x_i^{(p)}}^* \oplus \overline{x_i^{(q)}}^*$

... ... ... ... ... ... ... ... ...

(2) D-layer calculus theory :

From the definition 3 it is can be proved that.

(1a)' $x_i^{(s)} + x_j^{(t)} = x_j^{(t)} + x_i^{(s)}$

(1b)' $x_i^{(s)} . x_j^{(t)} = x_j^{(t)} . x_i^{(s)}$

(2a)' $x_h^{(r)} + ( x_i^{(s)} . x_j^{(t)} ) = ( x_h^{(r)} + x_i^{(s)} ) . ( x_h^{(r)} + x_j^{(t)} )$

(2b)' $x_h^{(r)} . ( x_i^{(s)} + x_j^{(t)} ) = ( x_h^{(r)} . x_i^{(s)} ) + ( x_h^{(r)} . x_j^{(t)} )$

(3a)' $x_i^{(s)} + \overline{x}_i^{(s)} \neq 1$

(3b)' $x_i^{(s)} . \overline{x}_i^{(s)} \neq 0$

(4a)' $\overline{x_i^{(s)} + x_j^{(t)}} = \overline{x}_i^{(s)} . \overline{x}_j^{(t)}$

(4b)' $\overline{x_i^{(s)} . x_j^{(t)}} = \overline{x}_i^{(s)} + \overline{x}_j^{(t)}$.

... ... ... ... ... ... ... ... ... ... ...

The following theorem can be obtained :

**Theorem 2.** $< \mathscr{X} , . , + , - , 0 , 1 >$ form a fuzzy De.Morgan algebra.

(3) **Conclusion :**

Based on the B-D type calculus theory $< \mathscr{P} , \cdot(\odot) , +(\oplus) , -(-^*) , 0 (0_i) , 1 ( 1_i ) >$ algebraic system is a multi-layer fuzzy B-D type algebraic system, a hard-and-soft algebraic system ;it has become the algebraic model of the multi-layer logical structure in the above system.

In this algebraic model, based on the above calculus characters worked out, the given B-D type fuzzy formula (function) can be turn into the standard form and simplified, also the true-false value of the language can be estimated, therefore the logical analysis on the complex system can be better carried out.

## REFERENCES


1. Zadeh,L.A., Fuzzy sets, Information and control. 8(1965),338-353.
2. Zadeh,L.A., Fuzzy sets as a basis for a theory of possibility, Fuzzy sets and systems 1(1978), 3-28.
3. Zadeh,L.A., Fuzzy logic as a basis for the management of uncertainty in expert systems Fuzzy sets and systems 11 (1983c), 199-227.
4. Negoita,C.V., and Ralescu, D.A., Applications of Fuzzy sets to systems analysis, 1975.
5. Dubois,D. and prade, H.(1980), Fuzzy sets and systems, theory and applications.Academic press. Inc., New York.
6. S.C.Kleene, Introduction to Metamathematics,p.334,Van Nostrand, New York,(1952).