

GENERALIZED PROBABILISTIC INDEPENDENCE AND UTILITY FUNCTIONS

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Abstract : The concept of probabilistic independence is generalized into a decomposability criterion by means of a two-place operation on the unit interval. The set of candidate operations is characterized via natural axioms owing to a recent result in the theory of functional equations. This result is applied to the construction of utility functions, by relaxing Von Neumann and Morgenstern's axiom system.

I - INTRODUCTION

For many authors, the uncertainty pervading decisions can be represented by probability measures on the set of possible consequences of these decisions. Justifications of the probabilistic setting are based on the betting behavior interpretation or qualitative probability relations [5]. Under the subjectivist point of view the concept of probabilistic independence is sometimes difficult to justify. This is because independence is understood, in the setting of statistics, between experiments. But in the subjectivist setting only independence between events can be defined, and the usual definition :

$$P(A \cap B) = P(A) \cdot P(B) \quad (1)$$

where A, B are subsets of Ω , and P is a probability measure, may appear too restrictive outside a frequentist framework.

A qualitative view of independence is developed in Fine [2] which leads to a decomposability criterion. Namely, A and B are separable if and only if there is an operation $*$ on $[0,1]$ such that

$$P(A \cap B) = P(A) * P(B) \quad (2)$$

i.e. $P(A \cap B)$ can be computed from the knowledge of $P(A)$ and $P(B)$. In the following, we study the possible candidates for operation $*$, and come up with a parametered family of semi-groups on $[0,1]$. On such a basis the question of constructing utility functions is reconsidered.

2 - SEPARABILITY OF JOINT EVENTS

The following requirements are natural for operation $*$:

- i) commutativity
- ii) associativity
- iii) $\forall a \in [0,1], a * 1 = a, 0 * 0 = 0$
- iv) $\forall a, b, c, d \in [0,1] \quad a \geq b, c \geq d \Rightarrow a * c \geq b * d$
- v) continuity
- vi) $\forall a \in (0,1) \quad a * a < a$

Commutativity stems from $A \cap B = B \cap A$. Associativity appears if one considers separability of n events $A_1 \dots A_n : \forall m \leq n, \forall k_1 \dots k_m \leq n,$

$$P(A_{k_1} \cap A_{k_2} \cap \dots \cap A_{k_m}) = P(A_{k_1}) * P(A_{k_2}) * \dots * P(A_{k_m}) \quad (3)$$

Associativity of $*$ is due to that of set-intersection. Axiom iii) is due to $A \cap \Omega = A, \emptyset \cap \emptyset = \emptyset$, and the fact that A and Ω are independent. iv) is natural if we interpret $a = P(A), b = P(B), c = P(C), d = P(D)$ and $A \subseteq B, C \subseteq D$, where A and B, C and D are separable, respectively. Continuity is also a natural technical requirement.

Under i)-iv), operation $*$ cannot be chosen but among triangular norms [6], which are semi-group operations on the unit interval, monotonically increasing in both places, with identity 1. Triangular norms are all such that

$$T_W(a,b) \leq a*b \leq \min(a,b) \quad (4)$$

where $T_W(a,b) = 0 \forall a, b \in [0,1)$, $T_W(a,1) = T_W(1,a) = a$.

The main continuous triangular norms, apart from \min and T_W are the product and $a*b = \max(0, a+b-1)$, also denoted T_m .

Axiom vi) is to account for the situation when $P(A) = P(B) = a$, $P(A-B) \neq 0$, $P(B-A) \neq 0$. Then clearly $P(A \cap B) < a$. The separability of A and B leads to admit axiom vi). A triangular norm satisfying i)-vi) is called Archimedean and is pseudo-additive in the sense that for there is a continuous, strictly decreasing mapping $f : [0,1] \rightarrow [0,+\infty)$ such that $f(1) = 0$, and (see [6])

$$f(a*b) = f(a) + f(b) \quad (5)$$

Operation $*$ cannot be arbitrarily chosen among triangular norms. First note that $\forall A, B$

$$\max(0, P(A)+P(B)-1) \leq P(A \cap B) \quad (6)$$

The equality (T_m -separability) is valid when A and B simultaneously occur as seldom as possible. With a subjectivist point of view, the equality expresses a pessimistic opinion on the simultaneous occurrence of A and B knowing $P(A)$ and $P(B)$.

On the other hand, $P(A \cap B)$ is bounded from above by $\min(P(A), P(B))$. The minimum operation violates axiom vi). Indeed $P(A \cap B) = \min(P(A), P(B))$ expresses a strong dependence between A and B : if $P(A) \geq P(B)$, it means that A occurs at least wherever B occurs; it expresses an optimistic opinion about the joint occurrence of A and B since it amounts to claiming that if the least probable event occurs then the other event should occur too. Hence $*$ -separability ranges from maximal exclusiveness (T_m -separability) to strong dependence (\min -separability). The usual independence ($*$ = product) expresses a moderate opinion between these extreme cases.

3 - CHARACTERIZATION OF EVENTS SEPARABILITY

So far, we have not used the additivity law of probability measures. We use it now in the following :

Lemma 1 : if A and B are \star -separable then there is an operation \circ such that \bar{A} and \bar{B} are \circ -separable.

$$\begin{aligned} \text{Proof : } P(\bar{A} \cap \bar{B}) &= 1 - P(A) - P(B) + P(A) \star P(B) \\ &= P(\bar{A}) + P(\bar{B}) - 1 + (1 - P(\bar{A})) \star (1 - P(\bar{B})) \\ &\stackrel{\Delta}{=} P(\bar{A}) \circ P(\bar{B}). \end{aligned} \quad \text{Q.E.D.}$$

As a consequence, operation \circ also belongs to Archimedean triangular norms, and is thus associative. Associativity of operation \circ implies associativity of operation $a+b = a \star b$, given that \star is associative. This property is not self-evident and severely restricts the choice of \star among Archimedean triangular-norms as shown by Frank [3]. Indeed, he has shown that the only Archimedean continuous triangular norms such that $a \star b$ and $a+b = a \star b$ are simultaneously associative are defined by

$$a \star b = a F_s b = \text{Log}_s \left[1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right] \quad s > 0 \quad (7)$$

For $s = 0, 1, +\infty$ we respectively recover $a \star b = \max(0, a+b-1)$, $a.b$, and $\min(a, b)$. Note that \star is strictly monotonic and satisfies (5) with $f_s(a) = \text{Log}_s \frac{s-1}{s^a-1}$, for $s \neq 0, 1, +\infty$. Now it is easy to check that if A and B are \star -separable for $\star = F_s$ then

$$\begin{aligned} &\bullet \bar{A} \text{ and } \bar{B} \text{ are } F_s \text{-separable, i.e. } P(\bar{A} \cap \bar{B}) = P(\bar{A}) F_s P(\bar{B}) \\ &\bullet A \text{ and } \bar{B}, \text{ as well as } \bar{A} \text{ and } B, \text{ are } F_{s'} \text{-separable with } s' = 1/s \\ &\text{i.e. } P(A \cap B) = P(A) - P(A \cap \bar{B}) = P(A) - P(A) F_s P(B) \\ &\quad = P(A) F_{1/s} P(B) \end{aligned}$$

For instance $P(A \cap B) = \min(P(A), P(B)) \Rightarrow P(A \cap \bar{B}) = \max(0, P(A) + P(B) - 1)$.
 \bullet A and B as well as \bar{A} and \bar{B} are both \star -separable if and only if they are both independent ($\star =$ product).

N.B. The quantity $a+b = a F_s b$ is equal to $1 - (1-a)F_s(1-b)$, and is called a triangular conorm.

4 - EXTENSION OF Von NEUMANN-MORGENSTERN AXIOMS OF UTILITY

Let X be a set containing the possible consequences of potential decisions. A set \underline{X} of uncertain consequences is built over X , and is such that

$$X \subseteq \underline{X} ; x, y \in X \Rightarrow (x, p, y) \in \underline{X}$$

where (x, p, y) is short for the uncertain consequence where x occurs with probability p and y with probability $1-p$. The choice among potential decisions is guided by a preference ordering \geq over uncertain consequences. \geq is supposed to be a weak order [4], i.e. is transitive, reflexive, and $\forall x, y \in \underline{X}$, $x \geq y$ or $y \geq x$. $>$ and \sim respectively denote the strict ordering and equivalence constructed from \geq . Notice that :

$$(x, 1, y) \sim x ; (x, p, y) \sim (y, 1-p, x).$$

The fundamental axioms of utility theory are then

$$U1) \quad x \sim y \rightarrow \forall p \in [0, 1], \forall z \in X \quad (x, p, z) \sim (y, p, z).$$

$$U2) \quad x > y \rightarrow \forall p \in (0, 1), \quad x > (x, p, y) > y.$$

$$U3) \quad x > y > z \Rightarrow \exists p \in]0, 1[\quad y \sim (x, p, z)$$

$$U4) \quad ((x, p, y), q, y) \sim (x, pq, y).$$

Given these axioms, it is proved in [7] that a utility function $u : \underline{X} \rightarrow \mathbb{R}$ exists such that $x > y$ iff $u(x) > u(y)$, $x \sim y$ iff $u(x) = u(y)$, and

$$u(x, p, y) = p u(x) + (1-p)u(y) \tag{8}$$

Moreover, u is an interval scale [4].

Here we shall focus on axiom U4, which the authors of [7] wished to modify, but, to them, "the mathematical difficulties seem to be considerable" (p. 632). The uncertain consequence $((x, p, y), q, y)$ can be viewed as the result of two successive decisions ; the first one enables the second one with probability q , and the second one leads to consequence x with probability p . On the whole, in this 2-stage decision process, x is got with probability $p.q$ and y with probability $1-pq$, as long as the events following the decisions are independent. Whether this assumption is verified is sometimes up to the decider's judgment.

He may consider on the contrary, that a favorable issue to both decisions (i.e. yielding x) is very unlikely, and then only grants a probability

$\max(p+q-1, 0)$ to x . Or he is very optimistic and admits that if the most risky decision reaches its goal, then so will the other ones, and so he grants probability $\min(p, q)$ to x . More generally, he takes for granted that $P(x) = p \star q$ where \star is a Frank triangular-norm. Hence we relax U4 into

$$U4') ((x, p, y), q, y) \sim (x, p \star q, y)$$

5 - CONSTRUCTION OF GENERALIZED UTILITY FUNCTIONS

Following the same reasoning process as in [7], we can prove the following results

Lemma 2 : $\forall p, q \in (0, 1), \forall x, y \in \underline{X}$,

$$x > y, p > q \Rightarrow (x, p, y) > (x, q, y).$$

Proof : Applying (U2) : $x > y \Rightarrow (x, p, y) > y \Rightarrow \forall r \in (0, 1), (x, p, y) > ((x, p, y)r, y)$
 $= (x, p \star r, q)$ due to (U4'). it is now enough to solve equation $p \star r = q$.

Using (5), r is unique and is equal to $f_s^{-1}(f_s(q) - f_s(p))$ for $s \neq +\infty, 1, 0$;
 $r = q$ if $\star = \min$; $r = q/p$ if $\star = \text{product}$; $r = 1 - p + q$ if $\star = T_m$. Q.E.D

Lemma 3 : $\forall p, q \in (0, 1) \forall x \neq y \in \underline{X}$, $x > y$ and $(x, p, y) \sim (x, q, y)$ imply $p = q$.

Proof : Obvious with Lemma 2.

Lemma 4 : $\forall q$, if $p > r$ and $x > y$ then there is α such that

$$((x, p, y), q, (x, r, y)) \sim (x, \alpha, y), \alpha \in [0, 1].$$

Proof : $x > (x, p, y) > (x, r, y) > y$, and there is some t such that $(x, p, y) \sim (x, t(x, r, y))$ using Lemma 3.

Now $(x, t(x, r, y)) \sim (x, t+r-t\star r, y)$, with $t+r-t\star r = 1 - (1-r)\star(1-t)$

since \star is a Frank t-norm. Hence we must find t such that

$$(1-r)\star(1-t) = 1-p$$

which yields $t = 1 - f_s^{-1}(f_s(1-p) - f_s(1-r)) \stackrel{\Delta}{=} p \ominus r$. This is a kind of subtraction since $t = 0$ for $r = p$. Now it is easy to check that

$$\begin{aligned}
((x, p, y), q, (x, r, y)) &\sim ((x, p\theta r, (x, r, y)), q, (x, r, y)) \\
&\sim (x, (p\theta r)*q, (x, r, y)) \quad (U4') \\
&\sim (x, (p\theta r)*q+r-(r\theta p)*q*r, y)
\end{aligned}$$

$$\text{Hence } \alpha = (p\theta r)*r+r-(p\theta r)*q*r \quad (9)$$

It is unique due to lemma 3.

Q.E.D.

Now if $p < r$, using $(x, p, y) \sim (y, 1-p, x)$, we come up with the same expression for α , where q is changed into $1-q$, r into p and conversely.

For $s = 1$, i.e. $*$ = product we get $\alpha = pq + (1-p)r \forall p, r$.

This is the usual assumption of independence, and it suggests the usual form of utility.

For $s = 0$, i.e. $a*b = \max(a+b-1, 0)$, we get $\forall p, r$

$$\alpha = \min(\max(r, q+p-1), \max(p, r-q)) \quad (10)$$

this is the pessimistic case.

For $s = +\infty$ i.e. $a*b = \min(a, b)$ we get, when $p > r$, $\alpha = \text{med}(p, q, r)$ where med stands for median, it is noticeable that the median is found here instead of an expectation-like expression (a weighted mean) in the usual case. A median is indeed the qualitative counterpart of a mean. More generally for $*$ = min we get

$$\alpha = \min(\text{med}(p, q, r), \text{med}(p, 1-q, r)) \quad (11)$$

Note that when $q = 0$, $\alpha = r$, and when $q = 1$, $\alpha = p$, so that α moves from r to p when q moves from 0 to 1. Actually, for $0 < s < +\infty$, α is strictly increasing with q when $p > r$.

In the following let us assume that \underline{X} contains a maximal element x_1 and a minimum element x_0 . We introduce a function $u_s : \underline{X} \rightarrow [0, 1]$ with $u_s(x_0) = 0$, $u_s(x_1) = 1$, $u_s(x) = q$ as soon as $x = (x_1, q, x_0)$. Parameter s is that of the separability operation $*$.

The following result is obtained :

Theorem 1 : $\forall s > 0, s < +\infty$, then

$$\forall x > y, u_s(x) > u_s(y) \quad (12)$$

$$\forall x > y, \forall q \in [0,1]$$

$$u_s(x, q, y) = (u_s(x) \ominus u_s(y)) * q + u_s(y) - (u_s(x) \ominus u_s(y)) * q + u_s(y) \quad (13)$$

Proof : (12) is obvious by using $x = (x_1, p, x_0)$ $y = (x_1, q, x_0)$.

(13) is obvious using the expression of α from Lemma 3.

Q.E.D

for $s = 1$ we get the usual utility function (8).

Now changing the scale $[0,1]$ into $I = [a_0, a_1]$ comes down to changing operations $\ominus, *$, etc... into \ominus_I in I , such that $\forall a, b \in [a_0, a_1]$,

$$a \ominus_I b = a_0 + (a_1 - a_0) \left[\frac{a - a_0}{a_1 - a_0} \ominus \frac{b - a_0}{a_1 - a_0} \right]$$

and utility function $u_s(x)$ into $u_s(x, I) = (a_1 - a_0)u_s(x) + a_0$; $u_s(., I)$ also satisfies (12) and a generalized form of (13), changing $\ominus, *$ and q into $\ominus_I, *_I$ and $a_1 q + a_0(1-q)$ respectively. We can also prove a unicity theorem :

Theorem 2 : if $u_s(., I)$ and $u_s(., I') = u'_s$ satisfy the generalized form of (13), and also (12), then $\forall \alpha, \beta : u_s(., I) = \alpha u_s(., I') + \beta$.

Proof : there is an increasing bijection f between u and u' such that

$$f(a_0) = a'_0, f(a_1) \text{ and } u' = f \circ u. \text{ Particularly}$$

$$\begin{aligned} f(u_s(x_1, q, x_0)) &= f((a_1 - a_0)q + a_0) \\ &= u'_s(x_1, q, x_0) = [f(a_1) - f(a_0)]q + f(a_0) \end{aligned}$$

$$\text{Hence } f \text{ is of the form } f(a) = \alpha a + \beta \quad \forall a \in I.$$

Q.E.D

Hence, $u_s(., I)$ is an interval scale.

5 - DISCUSSION

These results indicate that it is possible to build utility functions without the independence assumption, only assuming a separability condition. Note that theorems 1 and 2 are valid only for $0 < s < +\infty$. Indeed for $s = 0$ and $s = +\infty$, (12) does not hold i.e. we can have $x > y$ and $u(x) = u(y)$.

This occurs

- for $s = +\infty$, $x = (x_1, p, x_0) > y = (x_1, r, x_0)$ and $0 < p < \min(p, r) = r$
 - for $s = 0$, with the same conventions and $0 < q < 1 + r - p$,
- we then observe : $u_s(x, q, y) = r = u_s(y)$, while $(x, q, y) > y$.

In such cases, one may wish to relax axiom U2, changing $>$ into \geq . This behavior of the utility function means that if x becomes very unlikely (q is small enough), the utility of (x, q, y) cannot be distinguished from that of y (with certainty). Of course this attempt to extend the concept of utility could be considered in the situation when \underline{X} has no maximal nor minimal elements. Moreover, in this approach we have taken for granted the existence of probabilities of consequences ; modern approaches (e.g. [4]) construct both probability and utility functions from purely qualitative arguments. However, as Fine [2] pointed out, to justify additive probability requires the use of complicated or un-natural axioms. A more intuitive setting for extending utility theory in the spirit of separability assumptions may be that of decomposable measures [1] where the additivity axiom of probability measures is generalized by means of a triangular co-norm [6].

REFERENCES

- [1] - Dubois, D., Prade, H. A class of fuzzy measures based on triangular norms. Int. J. General Systems 8, 43-61, 1982.
- [2] - Fine, T. Theories of Probability. Academic Press, 1973.
- [3] - Frank, M.J. On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. Aequationes Math. 19, 194-226, 1979.
- [4] - Krantz, D.H., Luce, R.D., Suppes, S., Tversky, A. Foundations of Measurement. Vol. 1, Academic Press, 1971.
- [5] - Savage, L.J. The Foundations of Statistics, Dover, 1972.
- [6] - Schweizer, B., Sklar, A. Probabilistic Metric Spaces. North-Holland, 1983.
- [7] - Von Neuman, J., Morgenstern, O. Theory of Games and Economic Behavior, Princeton, Univ. Press, 1953.