The Weight-of-Conflict Conjecture

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In his book[1], Glenn Shafer proposes a conjecture, called the the Weight-of-conflict conjecture. It says, if the commonality functions Q_1 and Q_2 of two belief functions satisfy the inequality $Q_1(A) \leq Q_2(A)$ for all $A \subseteq \Theta$, then their weights of internal conflict W_{Q_1} and W_{Q_2} obey $W_{Q_1} \geq W_{Q_2}$.

The conjecture arises from an attempt to define the weight of conflict for support functions (see shafer [1]). In this paper the author gives out an alternative proposition, which solves the problem all the same.

1. Prerequisites

In this section, the author relates the basic concepts and results necessary to the paper. see safer[1] for details.

Let Θ be a finite set, call it a frame of discernment. A function $Bel: 2^{\Theta} \longrightarrow [0, 1]$ is a belief function if:

(1). $Bel(\emptyset)=0$, $Bel(\Theta)=1$;

(2). For each integer n and arbitrary subsets A_1, $A_2, \ldots A_n$ of Θ

$$Bel(\bigcup_{i=1}^{n} A_i) \geq \sum_{k=1}^{n} (-1)^{k-1} \sum \left\{ Bel(\bigcap_{i\in I} A_i) \;\middle|\; |I|=k,\ I \subseteq \{1,2, \cdots, n\} \right\} \tag{1.1}$$

Given a belief function Bel, there exists uniquely a map $M: 2^{\Theta} \longrightarrow [0, 1]$, called the basic probability assignment of Bel, such that:

$$Bel(A) = \sum \left\{ M(B) \;\middle|\; B \subseteq A \right\} \tag{1.2}$$

for every $A \in 2^{\Theta}$. The function $Q: 2^{\Theta} \longrightarrow [0, 1]$ defined by

$$Q(A) = \sum \left\{ M(B) \;\middle|\; B \supseteq A \right\} \tag{1.3}$$

for every $A \in 2^{\Theta}$, is called the commonality function of Bel.

For a pair of belief functions $\mathbf{Bel}_1$ and $\mathbf{Bel}_2$, denote their basic probablity assignments by M_1 and M_2 respectively. If the number

$$k = \sum\{M_1(A_1)\cdot M_2(A_2) \mid A_1, A_2 \in 2^{\Theta}, A_1 \cap A_2 \neq \emptyset\} \tag{1.4}$$

is not zero, then we say that the orthogonal sum of $\mathbf{Bel}_1$ and $\mathbf{Bel}_2$ exists. Denoted by $\mathbf{Bel}_1 \oplus \mathbf{Bel}_2$ the sum is defined to be the belief function whose basic probability assignment is given by

$$M(A) = \frac{1}{k}\sum\{M_1(A_1)\cdot M_2(A_2) \mid A_1 \cap A_2 = A\} \tag{1.5}$$

The number $-\log(\mathbf{k})$ is then called the weight of internal conflict of $\mathbf{Bel}_1$ and $\mathbf{Bel}_2$ and is denoted by $\mathbf{Con}(\mathbf{Bel}_1, \mathbf{Bel}_2)$. The orthogonal sum and weight of internal conflict of more than two belief functions are defined inductively. It can be showed that those definitions are in dependent of the order of combination, and that there exists the equality:

$$\mathbf{Con}(\mathbf{Bel}_1, \cdots\cdots, \mathbf{Bel}_n) = \mathbf{Con}(\mathbf{Bel}_1, \cdots\cdots, \mathbf{Bel}_{n-1}) + \mathbf{Con}(\mathbf{Bel}_1 \oplus \cdots \oplus \mathbf{Bel}_{n-1}, \mathbf{Bel}_n) \tag{1.6}$$

where $\mathbf{Con}(\mathbf{Bel}_1, \cdots\cdots, \mathbf{Bel}_n)$ stands for the internal conflict of $\mathbf{Bel}_1, \cdots\cdots. \mathbf{Bel}_n$; and the other items are similarly defined.

A focal element $\mathbf{A}$ of a belief function $\mathbf{Bel}$ over Θ is a subset of Θ such that the basic probability assignment of $\mathbf{Bel}$ assumes positive value to $\mathbf{A}$. The belief functions with possibly one focal element other than Θ are called simple support functions, and the belief functions which can be expressed as orthogonal sums of simple support functions are called support functions.

Suppose Θ and Ω are two frames of discernment, if there is a map $\omega: \Theta \longrightarrow 2^{\Omega}$ such that $\{\omega(\theta) \mid \theta \in \Theta\}$ constitues a partition for Ω, then we call Ω a refinement of Θ, and the map $\omega: \Theta \longrightarrow 2^{\Omega}$ a refining.

A belief function $\mathbf{Bel}$ over Θ is called a support function, if there is a refinement Ω of Θ and a separable support function $\mathbf{Bel}_0$ over Ω such that $\mathbf{Bel}_0 | 2^{\Theta} = \mathbf{Bel}$.

.2. The problem and its Solution

Our task is to define the weight of conflict for support functions. A obivous way of doing this is to proceed like this: first, for a given support function $\mathbf{Bel}$ over a frame of discernment Θ, construct set

$$\mathcal{W}_{Bel} = \{\mathbf{Bel}_0 \mid \mathbf{Bel}_0 \text{ is a separable support function over some refinement } \Omega \text{ of } \Theta \text{, satisfying } \mathbf{Bel}_0 | 2^{\Theta} = \mathbf{Bel}\} \tag{1.1}$$

then define:

$$W_{Bel} = \inf\{W_{Bel_0} \mid \mathbf{Bel}_0 \in \mathcal{W}_{Bel}\} \tag{2.2}$$

where W_{Bel_0} is the weight of conflict among among $\mathbf{Bel}_0$'s components.

But one matter to be consider here is that, provided that **Bel** itself is a separable support function, then whether the number defined by the right hand of (2.2) is equal to W_{Bel}. If not the definition is self-contadicting and therefore must be abandoned. It is aimed to justify (2.2) that G.Shafer brings forwards the Weight-of-Conflict conjecture. So, as its name suggests, the Weight-of-Conflict is essentialy about the adequance of (2.2). If the conjecture is true, (2.2) is indeed justified. But one can't say that's the only way to approch the problem. In fact, there are always a dozen of paths to the top of a hill.

Now, I would like to give out an alternative proposition, whose proof is given in next section, and explain its ability of fullfilling the job in hand.

Theorem: Suppose **S** and **T** are two separable support functions over a frame of discernment Θ, and let $\mathcal{M}_S$ be the ring generated by the focal elements of **S**. If $S(A) = T(A)$ for each $A \in \mathcal{M}_S$, then :

$$W_s \leq W_t \quad . \qquad (2.3)$$

Corollary: If **S** is a separable support function over Θ, then

$$W_S = \inf\{ W_{S_0} \mid S_0 \in \mathcal{W}_S \} \qquad (2.4)$$

Proof: By definition, each $S_0 \in \mathcal{W}_S$ is a separable support function over some refinement Ω of Θ such that

$$S_0 \mid 2^{\Theta} = S \; ; \qquad (2.5)$$

Let the refining is $\omega: \Theta \to 2^{\Omega}$, extend it to be a map from 2^{Θ} to 2^{Ω} by assigning every element **A** of 2^{Θ} the element $\bigcup\{ \omega(\theta) \mid \theta \in A \}$ of 2^{Ω}, and define map

$$S' : \; 2^{\Omega} \longrightarrow [0 \; , \; 1] \qquad (2.6)$$

$$B \longmapsto \sup\{ S(A) \mid \omega(A) \subseteq B \} \; .$$

It's easy to prove that **S'** is also separable support function and $W_{S'} = W_S$.

On the other hand, we can derive from (2.5) and (2.6) that $S'(B) = S_0(B)$ for each $B \in \mathcal{M}_{S'}$ Therfore $W_{S'} \leq W_{S_0}$ i.e $W_S \leq W_{S_0}$. Then (2.4) follows.

3. The Proof of the theorem

Proof: Let $S = S_1 \oplus \cdots \oplus S_n$, where S_i's are simple support functions with focal elements A_i's. We will prove the theorem by induction on **n**.

When **n** = 1, (2.3) is trivial.

Assume (2.3) is valid for all **k** less than **n**, now we set out to show the inequality for the case of **n**.

Choose from A_i's a A_{i_0}, say A_1, which is different from Θ and $\emptyset$, and let S_0 be the simple support function with focal element A_1 and $S_0(A) = 1$. Then the orthogonal sums $S_0 \oplus S$ and $S_0 \oplus T$ satisfy the following conditions:

i. For each **A**

$$S_0 \oplus S(A) = S_0 \oplus T(A) \qquad (3.1)$$

In fact

$$S_0 \oplus S(A) = [S(A \cup A_1^c) - S(A_1^c)]/[1-S(A_1^c)]$$
$$= [T(A \cup A_1^c) - S(A_1^c)]/[1-T(A_1^c)]$$
$$= S_0 \oplus T(A) .$$

2. As belief functions over A_1, $S_0 \oplus S$ and $S_0 \oplus T$ are separable support functions and $S_0 \oplus S$ can be expressed as an orthogonal sum of less than **n** simple support functions. To see this, let us define S_i', for **i** ranging from 2 to **n**, to be the simple support function with focal element $A_i \cap A_1$ (*) and $S_i'(A_i \cap A_1) = S_i(A_i)$ if $A_i \not\supseteq A_1$, and the vacuous belief function if $A_i \supseteq A_1$. Thus, one has the relation

$$S_0 \oplus S = S_2' \oplus \cdots\cdots \oplus S_n' \qquad (3.2)$$

Actually, for every $A \in 2^{A_1}$

$$S_0 \oplus S(A) = \frac{1}{k} \sum \Big\{ \prod_{i \in I} M_i(A_i) \cdot \prod_{i \in I^c}(1 - M_i(A_i)) \Big| I \subseteq \{2, \cdots\cdots, n\}, \; \Phi \neq A_1 \cap \bigcap_{i \in I} A_i \subseteq A \Big\} \qquad (3.3)$$

where $I^c \triangleq \{2, \cdots\cdots, n\} \setminus I$ and

$$k = \sum \Big\{ \prod_{i \in I} M_i(A_i) \prod_{i \in I^c}(1 - M_i(A)) \Big| I \subseteq \{2, \cdots\cdots, n\}, A_1 \cap \bigcap_{i \in I} A_i \neq \Phi \Big\}$$
$$= \sum \Big\{ \prod_{i \in I} M_i'(A_i \cap A_1) \prod_{i \in I^c}(1 - M_i'(A_1 \cap A_i)) \Big| I \subseteq \{2, \cdots\cdots, n\}, \bigcap_{i \in I}(A_1 \cap A_i) \neq \Phi \Big\}$$
$$= k' .$$

Hence

$$S_0 \oplus S(A) = \frac{1}{k'} \sum \Big\{ \prod_{i \in I} M_i'(A_1 \cap A_i) \cdot \prod_{i \in I^c}(1 - M_i'(A_1 \cap A_i)) \Big| I \subseteq \{2, \cdots\cdots, n\}, \quad \Phi \neq \bigcap_{i \in I}(A_1 \cap A_i) \subseteq A \Big\}$$

Putting 1 and 2 together, we get another equality

$$S_0 \oplus S \big|_{\mathcal{M}_{S_0 \oplus S}} = S_0 \oplus T \big|_{\mathcal{M}_{S_0 \oplus S}}$$

Consequently, the assumption of induction has the right to say that

$$W(S_0 \oplus S) \leq W(S_0 \oplus T) .$$

But

$$W(S_0 \oplus S) = W_S + Con(S_0 , S) ,$$
$$W(S_0 \oplus T) = W_T + Con(S_0 , T) .$$

and

$$Con(S_0 , S) = -\log(1 - S(A_1^c))$$
$$= -\log(1 - T(A_1^c))$$
$$= Con(S_0 , T) .$$

Therefore $W_S \leq W_T$. The theorem is proved.

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*) $A_i \cap A_1$ is allowed to be empty. In fact, if $A_2 \cap A_1 = \Phi$ and $S_2(A_2) \neq 1$, then

$$S_2' \oplus \cdots \oplus S_n' = S_3' \oplus \cdots\cdots \oplus S_n' .$$

References:

[1]. Glenn Shafer. A Mathematical Theory of Evidence. Printen University Press. 1976.

[2]. Wang Pei-Zhuang. Fuzzy Sets and Falling Shadows of Random Sets (in chinese). Beijing NOrmal University Press. 1985.

[3]. Zhang lian-wen. A Theory of Belief Functions and Fuzzy Sets. Exposition Series on Fuzzy Mathematics (in chinese). Math. Dept. Beijing Normal University. 1985.

Abstract

In this paper, the author solves, in essence, the Weight-of-Conflict conjecture proposed by Glenn Shafer in his [1].