A UNIFIED APPROACH TO RANKING FUZZY SETS IN THE SAME SPACE

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Abstract In this paper we discuss a question related to important and still open problem of ranking fuzzy sets. This topic is closely tied to multiattribute decision-making in presence of fuzziness.It is pointed out fuzzy relation equations can be viewed as a suitable and constructive tool for the ranking procedures.

Introduction

A question of ranking fuzzy sets defined in the same universe of discourse appears to be of significant importance,especially in decision-making procedures.Benefits coming from the use of fuzzy sets as suitable for expressing vague human judgements imposed a new problem concerned with final ranking of the fuzzy sets representing alternatives put into account.Now we are at position to have at our disposal a great variety of methods of ranking fuzzy sets;we refer the reader to recent extensive survey provided by Bortolan and Degani[3].In fact, a list of algorithms suggested is a broad one,all of them are similar in sense of the mathematical tool applied.Namely:the fuzzy sets are ordered(viz.their membership functions are ordered)by means of a single numerical quantity.In more transparent way:all of them provide the user by a mapping "f" such that every fuzzy set representing the alternative is characterized by a real number,usually lying in [0,1] interval(an exception is the proposal formulated by Dubois and Prade [4] where four indices are discussed).All the algorithms are diverse in sense of the mapping used.This,in turn,implies the results of ranking are sometimes scattered over the whole interval.It is especially valid when the fuzzy sets put into ranking procedure are strongly "overlapped".The mentioned fact has been underlined in illustrative examples discussed in [3].And,moreover,one can be a bit distrustful because those results require special attention.Notice that the similar situation occurs in probabilistic-like procedures applied for ranking the alternatives in random environment.Nevertheless several dominance concepts are related each other,namely one draws another.Unfortunatelly,we cannot perform such the analysis for the ranking procedures with fuzzy sets.A critical review of the ranking algorithms based on fuzzy sets can be found in Kickert [7].

In the paper we present a unified approach toward ranking fuzzy sets in a general setting of fuzzy relation equations. We will point out the ranking algorithm introduced by Baas and Kwakernaak [1]may be formulated,and,furthermore generalized in the formal setting of the abovementioned equations.A characteristic feature clearly distinguishing it from the previous ones relies on its constructive character rather than any intuitive background.Empirical data set that appears in description of so-called prototype situations generate the fuzzy relation of the ranking method.

Ranking methods-few remarks

The problem of rating multiple aspect alternatives can be compactly formulated as follows.Let be a set of alternatives $\underline{A}=\{A_1,$

$A_2,\dots,A_n\}$ evaluated in light of "m "criteria, usually of a competitive character. A merit of the i-th alternative with regard to the j-th criterion is given by the rating R_{ij} usually treated as the fuzzy set. In sequel each criterion is weighted by its relative importance W_j, $j=1,2,\dots,m$ formulated linguistically (for instance: very important, more or less important, unimportant). Both R_{ij} and W_j are treated as fuzzy sets in $[0,1]$. It would be of interest to recall the method presented in [1]. By this way, the final ranking is based on fuzzification of weighted average [6]. Finally we get "n" fuzzy sets in $[0,1]$, $Z_1,Z_2,\dots,Z_n : [0,1] \to [0,1]$. Contrary to nonfuzzy ranking, now we are faced with the problem of expressing to which extent the i-th alternative is preferred. Let $I(i)$ denote the degree of preference, viz. the degree to which one can judge the i-th alternative as the best one. In [1] $I(i)$ is computed by sup-min composition, namely

$$I(i) = \sup_{z_1,z_2,\dots,z_n \in [0,1]} \left[Z_1(z_1) \wedge Z_2(z_2) \wedge \dots \wedge Z_n(z_n) \wedge R(z_1,z_2,\dots,z_n,i) \right] \tag{1}$$

$i=1,2,\dots,n$, where R stands for the fuzzy relation equal to

$$R(z_1,z_2,\dots,z_n,i) = \begin{cases} 1, \text{if } z_i \geq z_j \text{ for all } z_j, j=1,2,\dots,n \\ 0, \text{otherwise} \end{cases} \tag{2}$$

The higher $I(i)$, the more preferable the i-th alternative. We must stress here the choice of the fuzzy relation that aggregates the collection of the fuzzy sets $Z_1,Z_2,\dots,Z_n$ into the scalar index $I(i)$. Concisely speaking, the form of the fuzzy relation realized by (2) is one of possible ways leading to ordering the fuzzy sets. Another form of the fuzzy relation can be found in [2], where some extra parameters are introduced. The choice of the fuzzy relation R in the form (2) is more or less arbitrary.As underlined by Kickert [7] the ranking performed by (1)-(2) provides us by much fuzzier indication of preferability in comparison to the probabilistic methods. In extreme (cf. [3]) this fuzzy relation would cause a complete lack of discriminating property of the method, viz. several Z_i have the values of the corresponding $I(i)$ equal to 1.0.

Relational model for ranking the fuzzy sets

In this section we redefine the problem of the previous section. The proofs of the propositions used here are well-documented in existing literature [5][8][9][10]. Denote

$$I = [I(i_1)\ I(i_2)\ \dots\ I(i_n)]$$

a fuzzy set of the degrees of preference, where each coordinate, $I(i_k)$, $k=1,2,\dots,n$ is given by (1). It is evident that (1) is nothing but the fuzzy relation equation of the type,

$$I = Z_1 \circ Z_2 \circ \dots \circ Z_n \circ R \qquad (3)$$

with "o" standing for sup-min composition. A next step is to consider a generalized version of the equation with sup-t composition [9],

$$I = Z_1 \boxdot Z_2 \boxdot \dots \boxdot Z_n \boxdot R \qquad (4)$$

where "t" is sought as any t-norm. At the moment we will restrict ourselves to continuos t-norms-then some assumptions, mainly of technical nature, may be omitted .

A. Determination the fuzzy relation R

A central question that arises now is concentrated on determination the fuzzy relation R. For purpose of the ongoing estimation procedure, let us consider a family of K "prototype" decision situations expressed by the fuzzy sets $Z_1^{(k)}, Z_2^{(k)}, \dots, Z_n^{(k)}$ $k=1,2,\dots,K$, and the corresponding degrees of fuzzy preferences $I^{(k)}$. The are "prototype" or "schematic" in such the sense, the decision-maker is willing to assign the respective degrees of preferability to each of the collection of the above given fuzzy sets. Therefore, the fuzzy relation R is not given a priori, as in the methods studied previously, but has to be computed on the basis of this data set provided. By direct application of some results coming from theory of fuzzy relation equations [9] (assuming $\mathcal{R} = \bigcap_{k=1}^{K} \mathcal{R}_k \neq \emptyset$, $\mathcal{R}_k = \{ R \mid Z_1^{(k)} \square Z_2^{(k)} \square \ldots \square Z_n^{(k)} \square R = I^{(k)} \}$ we immediately obtain,

$$\hat{R} = \bigcap_{k=1}^{K} (\underline{Z}^{(k)} \circledcirc{\varphi} I^{(k)}) \qquad (5)$$

$$\hat{R}(z_1, z_2, \ldots, z_n, i) = \min_{1 \le k \le K} [(Z_1^{(k)}(z_1) t Z_2^{(k)}(z_2) t \ldots t Z_n^{(k)}(z_n)) \varphi I^{(k)}(i)]^{xx} \qquad (6)$$

as the greatest element of $\mathcal{R}$. It could happen the solution of the set of the equations for the data set does not exist. If this holds, an approximate solution of these equations may be searched.

B. On evaluation a discriminant power of the ranking procedure

Fuzzy relation equations can serve as a useful tool to give a deep insight into the structure of the algorithms induced by the fuzzy relation R. Let I, as before, be the fuzzy set containing various degrees of preference. Moreover, if all the fuzzy sets $Z_1, Z_2, \ldots, Z_{l-1}, Z_{l+1}, \ldots Z_n$, except one, Z_l, are fixed, then we may ask which is the greatest fuzzy set Z_l that does not change the fuzzy set I. In order words, we have to solve the equation

$$I = Z_1 \square Z_2 \square \ldots \square Z_{l-1} \square Z_l \square Z_{l+1} \square \ldots \square Z_n \square R \qquad (7)$$

with respect to Z_l. Also in this case the results of fuzzy relation equations enable us to calculate Z_l. Now we get

$$\hat{Z}_l = (Z_1 \square Z_2 \square \ldots \square Z_{l-1} \square Z_{l+1} \square \ldots \square Z_n \square R) \circledcirc{\varphi} I \qquad (8)$$

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xx φ is known as a so-called pseudocomplement ,

$$a \varphi b = \sup\{ c \in [0,1] \mid a t c \le b \} \quad a, b \in [0,1] .$$

as the greatest fuzzy set satisfying the given equation. Moreover by calculation lower solutions [5][8] we notice to which extent the fuzzy relation R discriminates the respective fuzzy sets Z_i. $\hat{Z}_1$ is the greatest element of $\mathcal{Z}_1 = \{Z_1 : [0,1] \to [0,1] \mid Z_1 \circ Z_2 \circ \ldots \circ Z_n \circ R = I\}$ while $Z_{11}, Z_{12}, \ldots, Z_{1p}$ are lower incomparable solutions of $\mathcal{Z}_1$; p stands for the number of lower solutions. This means $Z_{11}, Z_{12}, \ldots, Z_{1p}, \hat{Z}_1$ are indistinguishable viz. the alternatives producing each Z_{1j}, $j=1,2,\ldots,p$, $\hat{Z}_1$ lead to the same value of the grade of preference I.

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