The evolution of fuzziness and the direction of time

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A class of mathematically and empirically irreversible processes is studied. For analogy, if you know nothing about your old friend, then your uncertainty concerning his state naturally increases in the course of time

'The irreversibility that we observe is a feature of theories that take proper account of the nature and limitation of observation.' - [8, p. 215]

1. Problem

Let us consider the proposition 'There exist irreversible (one-directional) processes'. The problem of time direction consists in the fact that this proposition, being empirically certain, does not yet have nomological foundations generally acknowledged [10, c. 168--178]. The point is that the ordinary/partial differential equations of any physical theory permit the inversion of the temporal variable and, hence, can not by themselves single out the temporal direction. However, a decisive step seems to have already been made, by Prigogine.

In these notes let us show another particular approach to time direction. Here is its summary: a class of abstract irreversible processes, namely the class of irreversible solutions of 'fuzzy' Cauchy problems, is introduced; for interpretation and to verify the irreversibility, we refer to those dynamical systems our knowledge of the current states of which is marked by some fuzziness (incompleteness), with this fuzziness increasing as a continuous function of time. So a perceptive (and reflective) subject comes into being, and the approach proves to be anthropologistic in substance, and fuzzinistic in form, but not subjectivistic - for every present man ought to be that subject.

To illustrate what irreversibility will be studied, let the notation

$$\frac{d}{d\tau}x = u\,,\quad \tau\in R^1,\quad x\in R^n,\quad u\sim u_0\in R^n$$

represent a dynamical system under constant fuzzy control, implying that the control's current value is perceived as a fuzzy vector (a fuzzy point of R^n) independent of time and the system's dynamics. By x_τ and μ_τ we indicate the system's current states, dynamical and fuzzy, respectively (initiated at some moment of time). Also, by $\mathrm{fuzz}\,\mu_\tau$ we indicate the fuzziness ('radius') of μ_τ . Then, in the absence of real observations, μ_τ will represent our (un)knowledge of x_τ , with $\mathrm{fuzz}\,\mu_\tau$ increasing like some linear function of τ

2. Preliminaries

Let n be a positive integer. As usual, by R^n we mean the n-dimensional Euclidean space. Assuming also that R^1 is ordinarily ordered, it will be called the temporal axis. Let

$$x, y, x_H, \dot{x}\in R^n;$$
$$\tau, \tau_H, \tau_0, \tau_1\in R^1,\ \tau_0\le\tau_1;\ I_H=[\tau_H, \tau_H+1];$$
$$0\le\gamma<\infty;\ 0<\lambda<1;\ \sup\{\lambda:\lambda\in\emptyset\}=0.$$

By Ω we denote the classical space of non-empty convex compact subsets of R^n. In particular, Ω is supposed to be equipped with the Minkowski addition $+$, the positive homothetic and the Hausdorff metric $\rho(\cdot,\cdot)$ - see [7]. Being the operation without the inverse one, Minkowski's addition is not a standard addition. Hausdorff's metric is coordinated with the Minkowski addition and positive homothetic.

Let a family $\{\mathrm{lev}_\lambda \eta\}$ of the parameter λ be contained in Ω, be uniformly bounded and be non-increasing. We supplement it by the two members:

$$\mathrm{core}\,\eta = \bigcap_\lambda \mathrm{lev}_\lambda \eta, \quad \mathrm{supp}\,\eta = \overline{\bigcup_\lambda \mathrm{lev}_\lambda \eta}$$

(the line means closing in R^n). If $\mathrm{core}\,\eta$ consists of a single point, denoted by $\mathrm{argcore}\,\eta$, then the resulting family will be treated as a generalized point of R^n and be called the fuzzy point η, with

$$\mathrm{lev}_\lambda \eta, \quad \mathrm{core}\,\eta, \quad \mathrm{supp}\,\eta,$$

and the scalar-valued function

$$x \to \eta(x) = \sup\{\lambda : x \in \mathrm{lev}_\lambda \eta\}$$

being called its λ-level set, its core, support, and characteristic function. Cf. [9; 5]. In the case when $\mathrm{lev}_\lambda \eta \equiv \{y : y = x\}$ we say of the concentrated point δ_x as well. The collection of all the fuzzy points will be denoted by $\mathfrak{M}$. Of course, $\{\delta_x\}$ - the collection of all δ_x - will enter into $\mathfrak{M}$. Let

$$\mu, \nu, \xi, \mu_H, \zeta_0 \in \mathfrak{M}.$$

By means of Ω we define on $\mathfrak{M}$ the equality $=$, the Minkowski addition $+$, the positive homothetic, the Hausdorff metric $\rho(\cdot,\cdot)$, and the fuzziness fuzz:

$$\mu = \nu \iff \mathrm{lev}_\lambda \mu = \mathrm{lev}_\lambda \nu \quad \forall \lambda ,$$

$$\mathrm{lev}_\lambda (\mu + \nu) = \mathrm{lev}_\lambda \mu + \mathrm{lev}_\lambda \nu ,$$

$$\mathrm{lev}_\lambda \gamma \mu = \gamma \, \mathrm{lev}_\lambda \mu ,$$

$$\rho(\mu, \nu) = \sup_\lambda \rho(\mathrm{lev}_\lambda \mu, \mathrm{lev}_\lambda \nu) ,$$

$$\mathrm{fuzz}\, \mu = \rho(\mathrm{core}\, \mu, \mathrm{supp}\, \mu).$$

Functions from R^1 to R^n, Ω, and $\mathfrak{M}$ will be called vector, multi-valued, and fuzzy, respectively. Let $x_\cdot$ be a vector function, and $\mu_\cdot$ be a fuzzy one. In the case when $\mu_\tau \equiv \delta_{x_\tau}$ we say of the concentrated function $\delta_{x_\cdot}$ as well. The scalar function

$$\tau \to \mathrm{fuzz}\, \mu_\tau$$

will be named the fuzziness of $\mu_\cdot$. Following [3] and [1], let us introduce for multi-valued functions the differentiation $d/d\tau$ and the definite integral $\int_{\tau_0}^{\tau_1} \ldots d\tau$. By means of these structures we do the same for fuzzy functions:

$$\mathrm{lev}_\lambda \frac{d}{d\tau} \mu_\tau = \frac{\partial}{\partial \tau} \mathrm{lev}_\lambda \mu_\tau ,$$

$$\mathrm{lev}_\lambda \int_{\tau_0}^{\tau_1} \mu_\tau \, d\tau = \int_{\tau_0}^{\tau_1} \mathrm{lev}_\lambda \mu_\tau \, d\tau$$

(supposing the sets on the right exist and form the level set families of the fuzzy points under introduction).

The equivalent definitions based on the fundamental idea of duality can be found in [2], taking [10] as a guarantee of the duality.

Finally, let $f(\cdot)$ be a Lipschitzian mapping from R^n to R^n, and $\varphi(\cdot)$ from $(\mathfrak{M}, \rho)$ to $(\mathfrak{M}, \rho)$. In the case when $\varphi(\mu) \equiv \delta_{f(\operatorname{argcore}\mu)}$ we say of the mapping $\delta_f(\cdot)$ as well

3. Fuzzy dynamics versus reversibility

The notation

$$\frac{d}{d\tau}\mu = \varphi(\mu), \quad \mu_{\tau_H} = \mu_H$$

will be called the fuzzy Cauchy problem (φ, μ_H). Its solution should be thought of as a differentiable fuzzy function μ. such that

$$\frac{d}{d\tau}\mu_\tau = \varphi(\mu_\tau) \quad \forall \tau, \quad \mu_{\tau_H} = \mu_H .$$

The solution whose the fuzziness increases (strictly, as a scalar function) will be called irreversible.

Proposition. 1) The solution of each fuzzy Cauchy problem exists on I_H and is unique on it (and continuously depends on the problem, and can be found by the method of successive approximations),

2) the solution's fuzziness can not decrease,

3) the irreversible solutions exist.

Proof. 1) This follows from Banach's contraction principle over the standard scheme [6, p. 24--26]; the continuous dependence follows from Gronwall's inequality;

indeed,

$$\mu = \nu + \zeta \Rightarrow \mathrm{fuzz}\,\mu \geq \mathrm{fuzz}\,\nu$$

(for all μ, ν, ζ); if now $\mu_\cdot$ is the solution of (φ, μ_H), then $\int_{\tau_0}^{\tau_1} \varphi(\mu_\tau)\,d\tau$ exists, and

$$\mu_{\tau_1} = \mu_{\tau_0} + \int_{\tau_0}^{\tau_1} \varphi(\mu_\tau)\,d\tau \;;$$

hence, the fuzziness does not decrease;

[illegible]) put

$$\varphi(\mu) = \jmath_0 \,,\quad \mu_\tau = \delta_{x_H} + |\tau - \tau_H|\,\jmath_0 \;;$$

then $\mu_\cdot$ is the solution of (φ, δ_{x_H}) on I_H ; taking into account that

$$\mathrm{fuzz}\,(\delta_x + \mu) = \mathrm{fuzz}\,\mu \,,$$

$$\mathrm{fuzz}\,\gamma\mu = \gamma\,\mathrm{fuzz}\,\mu \,,$$

we derive

$$\mathrm{fuzz}\,\mu_\tau = |\tau - \tau_H|\,\mathrm{fuzz}\,\jmath_0 \;;$$

it remains to require $\jmath_0 \notin \{\delta_x\}$ because

$$\mathrm{fuzz}\,\mu = 0 \Leftrightarrow \mu \in \{\delta_x\} .$$

Another example: puting

$$\varphi(\mu) = \mu \,,\quad \mu_\tau = e^{\tau - \tau_H}\mu_H \,,\quad \mu_H \notin \{\delta_x\} ,$$

we find $\mu_\cdot$ to be the irreversible solution of (φ, μ_H), with

$$\mathrm{fuzz}\,\mu_\tau = e^{\tau - \tau_H}\,\mathrm{fuzz}\,\mu_H$$

4. The correspondence argument in support of reversibility

'Theoretical reversibility arises from ... idealizations ... that go beyond the possibilities of measurement performed with any finite precision.' – [8, p. 215]

The notation

$$\frac{d}{d\tau}x = f(x) \ , \ x_{\tau_H} = x_H$$

is called the vector Cauchy problem (f, x_H). Its solution is thought of as a differentiable vector function $x_.$ such that

$$\frac{d}{d\tau}x_\tau = f(x_\tau) \ \ \forall \tau \ , \ x_{\tau_H} = x_H .$$

This solution $x_.$ exists and is unique [6, p. 26]; furthermore, it permits reversing in the sense that the vector function $\tau \to x_{-(\tau-\tau_H)+\tau_H}$ also is the solution of a vector Cauchy problem, for instance, of $(-f, x_H)$.

In the case of the fuzzy Cauchy problem (δ_f, δ_{x_H}) we say of the concentrated Cauchy problem $\delta_{(f, x_H)}$ as well.

Proposition. The solution of each concentrated Cauchy problem is a concentrated function. In fact, when $x_.$ is the solution of (f, x_H), then $\delta_{x_.}$ is the solution of $\delta_{(f, x_H)}$.

The solution $\mu_.$ of a fuzzy Cauchy problem will be called reversible if the fuzzy function $\tau \to \mu_{-(\tau-\tau_H)+\tau_H}$ also is the solution of some fuzzy Cauchy problem. (Each reversible solution has the constant fuzziness, so all the irreversible solutions are non-reversible.)

Corollary. The solution of each concentrated Cauchy problem is reversible

5. Interpretation

On second thoughts, let us formulate the following postulates:

1) the temporal axis represents the time;

2) there exists a subject through the agency of which $\varphi(\cdot)$ describes a dynamical system under fuzziness, with μ and $\varphi(\mu)$ representing its admissible fuzzy state and the appropriate fuzzy velocity (more precisely, if μ represents a current fuzzy state of the system, then

$$x,\ \dot{x}$$

represent its current dynamical state and velocity, and

$$\mu(x),\ \varphi(\mu)(\dot{x})$$

serve as subjective evaluations of completeness of the last representation, so that $\operatorname{argcore}\mu$ and $\operatorname{argcore}\varphi(\mu)$ subjectively most completely represent its current dynamical state and velocity); the solution of (φ, μ_H) represents a fuzzy process in the mentioned system; the irreversible solutions represent the irreversible fuzzy processes;

3) every present man ought to be that subject.

Comment. According to [4], any proper physical theory must not involve perceptive subjects as its referents. This leads us to conclude that the fuzzy processes are not proper physical processes. In corroboration it can be found that the fundamental physical theories are 'dynamical' and 'statistical' [12] - but not 'fuzzy'

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