FUZZY RINGS IN THE SENSE OF SET-EMBEDDING

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Abstract

The purpose of this paper is to quote the method of the set-embedding expressing the fuzzy set, we open discussion into the fuzzy rings. And we prove some fundamental properties and theorems.

Keywords: Fuzzy Subring, Fuzzy Ideal.

1. Preliminaries

We first recall some basic concepts occurring in the papers [1, 2] for sake of completeness. In this paper X always denotes a nonempty (usual) set. P(X) will denote the power set of the set X. A fuzzy set in X is a map $\underset{\sim}{A}: X \to [0,1]$ and F(X) denotes the family of all fuzzy sets in X.

1. Set-Embedding: Let H is a map of [0,1] into P(X), $t \mapsto H(t)$. H is

called a set-embedding of X, if

$t_1 < t_2$ implies $H(t_1) \supseteq H(t_2)$.

Now let $\mathcal{U}(X)$ be the family of all set- embeddings of X. Let $\cup$, $\cap$, ^{c} be the operations in $\mathcal{U}(X)$, they are defined as follows:

$H_1 \cup H_2$: $(H_1 \cup H_2)(t) = H_1(t) \cup H_2(t)$

$H_1 \cap H_2$: $(H_1 \cap H_2)(t) = H_1(t) \cap H_2(t)$

H^c : $H^c(t) = (H(1 - t))^c$

$\bigcup_{r\in\Gamma} H_r$: $(\bigcup_{r\in\Gamma} H_r)(t) = \bigcup_{r\in\Gamma} H_r(t)$

$\bigcap_{r\in\Gamma} H_r$: $(\bigcap_{r\in\Gamma} H_r)(t) = \bigcap_{r\in\Gamma} H_r(t)$

2. Decomposition Theorem III. Let $\underset{\sim}{A} \in F(X)$, and $H: [0,1] \to P(X)$, $t \mapsto H(t)$, satisfies $A_{\underset{\cdot}{t}} \subseteq H(t) \subseteq A_t$, $t \in [0,1]$, then

1) $\underset{\sim}{A} = \bigcup_{t\in[0,1]} t\, H(t)$ (that is, $\underset{\sim}{A}(x) = \bigvee_{t\in[0,1]} (t \wedge H(t)(x))$, $H(t)(x)$ is a characteristic function $\chi_{H(t)}(x)$ of $H(t)$).

2) $t_1 < t_2$ implies $H(t_1) \supseteq H(t_2)$ (that is, $H \in \mathcal{U}(X)$).

3) $A_t = \bigcap_{\alpha<t} H(\alpha)$ $(t \in (0,1])$

$A_{\underset{\cdot}{t}} = \bigcup_{\alpha>t} H(\alpha)$ $(t \in [0,1))$

(The proof of the Decomposition Theorem cf [1])

3. Extension Principle: Let $f: X \to Y$, $x \mapsto f(x)$.

1) Extension Principle I: f can induce f: $F(X) \to F(Y)$,

$\underset{\sim}{A} \mapsto f(\underset{\sim}{A}) = \bigcup_{t\in[0,1]} t\, f(A_t) \in F(Y)$

$f^{-1}: \quad F(Y) \to F(X), \ \underset{\sim}{B} \mapsto f^{-1}(\underset{\sim}{B}) = \bigcup_{t\in[0,1]} t f^{-1}(B_t) \in F(X).$

(where $f(A_t) = \{y \mid \exists x \in A_t,\ y = f(x)\}$, $f^{-1}(B_t) = \{x \mid f(x) \in B_t\}$)

$f(\underset{\sim}{A})$ is called the image of $\underset{\sim}{A}$ and $f^{-1}(\underset{\sim}{B})$ is called the inverse image of $\underset{\sim}{B}$.

2) Extension Principle I : Let $\underset{\sim}{A} \in F(X)$, $f(\underset{\sim}{A}) = \bigcup_{t\in[0,1]} t f(A_{\underset{\cdot}{t}})$, and let $\underset{\sim}{B} \in F(Y)$, $f^{-1}(\underset{\sim}{B}) = \bigcup_{t\in[0,1]} t f(B_{\underset{\cdot}{t}})$.

3) Extension Principle II : If $\underset{\sim}{A} \in F(X)$, $A_{\underset{\cdot}{t}} \subseteq H_A(t) \subseteq A_t$, $t \in [0,1]$, then $f(\underset{\sim}{A}) = \bigcup_{t\in[0,1]} t\ f(H_A(t))$.

If $\underset{\sim}{B} \in F(X)$, $B_{\underset{\cdot}{t}} \subseteq H_B(t) \subseteq B_t$, $t \in [0,1]$, then

$$f^{-1}(\underset{\sim}{B}) = \bigcup_{t\in[0,1]} t\ f(H_B(t)).$$

4) The membership function of $f(\underset{\sim}{A})$ and $f^{-1}(\underset{\sim}{B})$ is $f(\underset{\sim}{A})(y) = \bigvee_{f(x)=y} \underset{\sim}{A}(x)$ and $f^{-1}(\underset{\sim}{B})(x) = \underset{\sim}{B}(f(x))$, respectively.

2. Fuzzy Subrings and Fuzzy Ideals

Let $(X,+,\cdot)$ is a ring. $R(X)$ and $I(X)$ will denote the set of all subrings of X and the set of all ideals of X, respectively. It is clear that $I(X) \subseteq R(X) \subseteq P(X)$. We will follow a convention that $\emptyset \in I(X) \subseteq R(X)$.

DEFINITION 1. Let $H \in \mathcal{U}(X)$. H is called a fuzzy subring of X, if $H(t) \in R(X)$ and $\underset{\sim}{H} = \bigcup_{t\in[0,1]} t\ H(t) \in F(X)$, $t \in [0,1]$.

$\underset{\sim}{R}(X)$ will denote the set of all fuzzy subrings of X.

Let $M \in \mathcal{U}(X)$. $\underset{\sim}{M}$ is called a fuzzy ideal of X, if

$M(t) \in I(X)$ and $\underset{\sim}{M} = \bigcup_{t\in[0,1]} t\, M(t) \in F(X)$, $t \in [0,1]$. $\underset{\sim}{I}(X)$ denotes the set of all fuzzy ideals of X.

The fuzzy subrings and the fuzzy ideals of X can be defined by the subring-embeddings and ideal-embeddings, respectively.

THEOREM 1 Let $(X,+,\cdot)$ be a ring, and let $\underset{\sim}{H}$, $\underset{\sim}{M} \in F(X)$, then:

1) $\underset{\sim}{H} \in \underset{\sim}{R}(X)$ iff, for $t \in [0,1]$, $H_t \in R(X)$.

2) $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ Iff, for $t \in [0,1]$, $M_t \in I(X)$.

Proof. 1) Let $\underset{\sim}{H} \in \underset{\sim}{R}(X)$, then there exists a subring-embedding $H = \{H(t) \mid t \in [0,1]\}$ such that $H(t) \in R(X)$, $t \in [0,1]$, and $\underset{\sim}{H} = \bigcup_{t\in[0,1]} t\, H(t)$. Therefore

$$H_t = \bigcap_{\alpha<t} H(\alpha) \in R(X), \quad t \in [0,1]$$

$$H_0 = X \in R(X).$$

Conversely, let $\underset{\sim}{H} \in F(X)$ satisfying $H_t \in R(X)$, $t \in [0,1]$, then $\{H_t \mid t \in [0,1]\}$ is a sub-ring-embedding . Therefore $\underset{\sim}{H} = \bigcup_{t\in[0,1]} t\, H_t \in R(X)$.

It can be seen in a similar way that 2) holds. //

THEOREM 2 Let $(X,+,\cdot)$ be a ring, and $\underset{\sim}{H}$, $\underset{\sim}{M} \in F(X)$, then:

1) $\underset{\sim}{H} \in \underset{\sim}{R}(X)$ iff, for any $x,y \in X$,

i) $\underset{\sim}{H}(x - y) \geqslant \underset{\sim}{H}(x) \wedge \underset{\sim}{H}(y)$;

ii) $\underset{\sim}{H}(xy) \geqslant \underset{\sim}{H}(x) \wedge \underset{\sim}{H}(y)$.

2) $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ iff, for any x, $y \in X$,

i) $\underset{\sim}{M}(x - y) \geqslant \underset{\sim}{M}(x) \wedge \underset{\sim}{M}(y)$;

ii) $\underset{\sim}{M}(xy) \geqslant \underset{\sim}{M}(x) \vee \underset{\sim}{M}(y)$.

Proof. 1). i) Let $\underset{\sim}{H} \in \underset{\sim}{R}(X)$, then $H_t \in R(X)$, $t \in [0,1]$. Let us assume $t = \underset{\sim}{H}(x) \wedge \underset{\sim}{H}(y)$. Then it follows $\underset{\sim}{H}(x) \geqslant t$, $\underset{\sim}{H}(y) \geqslant t$. So $x \in H_t$, $y \in H_t$. Thus $x - y \in H_t$. Therefore $\underset{\sim}{H}(x - y) \geqslant t = \underset{\sim}{H}(x) \wedge \underset{\sim}{H}(y)$.

It can be seen in a similar way that ii) holds.

Conversely, let $\underset{\sim}{H} \in F(X)$ satisfying i) and ii). Let $x, y \in H_t$, $t \in [0,1]$. Then $\underset{\sim}{H}(x) \geqslant t$, $\underset{\sim}{H}(y) \geqslant t$. So $\underset{\sim}{H}(x - y) \geqslant \underset{\sim}{H}(x) \wedge \underset{\sim}{H}(y) \geqslant t$ and $\underset{\sim}{H}(xy) \geqslant \underset{\sim}{H}(x) \wedge \underset{\sim}{H}(y) \geqslant t$. Thus $x - y$, $xy \in H_t$. Therefore $H_t \in R(X)$. Thus $\underset{\sim}{H} = \bigcup_{t \in [0,1]} t\, H_t \in \underset{\sim}{R}(x)$.

It can be seen in a similar way that 2) holds.

DEFINITION 2 Let $(X,+,\cdot)$ be a ring, and let $a \in X$. And let $H = \{H(t) \mid t \in [0,1]\} \in \mathcal{U}(X)$ and $\underset{\sim}{H} = \bigcup_{t \in [0,1]} t\, H(t) \in F(X)$. Then $(a+H)(t)$ is a residue class of X($(a + H)(t) = \{a + x \mid x \in H(t)\}$, $t \in [0,1]$).

$a + \underset{\sim}{H}$ is called a fuzzy residue class, if $a + \underset{\sim}{H} = \bigcup_{t \in [0,1]} t\, (a + H)(t)$.

THEOREM 3 Let $(X,+,\cdot)$ be a ring. And let $H \in \mathcal{U}(X)$ and $\underset{\sim}{H} = \bigcup_{t \in [0,1]} tH(t) \in F(X)$ and $a \in X$. Then

$$a + \underset{\sim}{H} = \bigcup_{t \in [0,1]} t\, (a + H)(t) = \bigcup_{t \in [0,1]} t\, (a + H_t).$$

Proof. $$\Big(\bigcup_{t \in [0,1]} t\, (a + H)(t)\Big)(x) = \bigvee_{t \in [0,1]} (t \wedge (a + H)(t)(x))$$
$$= \bigvee_{t \in [0,1]} \{t \mid x \in (a + H)(t)\}.$$

There exists a $y \in H(t)$ such that $x = a + y$ since $x \in (a + H)(t)$.

So $x - a = y \in H(t)$. Thus

$$\bigvee_{t\in[0,1]}\{t \mid x \in (a + H)(t)\} = \bigvee_{t\in[0,1]}\{t \mid x - a \in H(t)\}$$
$$= \bigvee_{t\in[0,1]}(t \wedge H(t)(x - a))$$
$$= (\bigcup_{t\in[0,1]} t\, H(t))(x - a)$$
$$= \underset{\sim}{H}(x - a);$$

$$(\bigcup_{t\in[0,1]} t\,(a + H_t))(x) = \bigvee_{t\in[0,1]}(t \wedge (a + H_t)(x))$$
$$= \bigvee_{t\in[0,1]}\{t \mid x \in a + H_t\}.$$

There exists a $y \in H_t$ such that $x = a + y$ since $x \in a + H_t$.

So $x - a = y \in H_t$. Thus

$$\bigvee_{t\in[0,1]}\{t \mid x \in a + H_t\} = \bigvee_{t\in[0,1]}\{t \mid x - a \in H_t\}$$
$$= \bigvee_{t\in[0,1]}(t \wedge H_t(x - a))$$
$$= (\bigcup_{t\in[0,1]} t\, H_t)(x - a)$$
$$= \underset{\sim}{H}(x - a). \qquad //$$

COROLLARY 1 Let $(X,+,\cdot)$ be a ring .And let $\underset{\sim}{H} \in F(X)$ and $a \in X$. Then $a + H$'s membership function $(a + \underset{\sim}{H})(x) = \underset{\sim}{H}(x - a)$. //

PROPOSITION 1 Let $(X,+,\cdot)$ be a ting. And let $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ and a, $b \in X$. Then

$$a + \underset{\sim}{M} = b + \underset{\sim}{M} \text{ iff } \underset{\sim}{M}(b - a) = \underset{\sim}{M}(0),$$

where 0 is the zero element of X.

Proof. Let us assume $a + \underset{\sim}{M} = b + \underset{\sim}{M}$, for any $x \in \underset{\sim}{X}$, we have $(a + \underset{\sim}{M})(x) = (b + \underset{\sim}{M})(x)$. $\underset{\sim}{M}(x - a) = \underset{\sim}{M}(x - b)$ by corollary 1 . Let $x = b$, then $\underset{\sim}{M}(b - a) = \underset{\sim}{M}(0)$.

Conversely, let us assume $\underset{\sim}{M}(b - a) = \underset{\sim}{M}(0)$, for any $x \in X$, we have

$$\begin{aligned} \underset{\sim}{M}(x - a) &= \underset{\sim}{M}(x - b + b - a) \geqslant \underset{\sim}{M}(x - b) \wedge \underset{\sim}{M}(b - a) \\ &= \underset{\sim}{M}(x - b) \wedge \underset{\sim}{M}(0) \\ &= \underset{\sim}{M}(x - b). \end{aligned}$$

It can be seen in a similar way that $\underset{\sim}{M}(x - b) \geqslant \underset{\sim}{M}(x - a)$. Thus $\underset{\sim}{M}(x - a) = \underset{\sim}{M}(x - b)$. Therefore $a + \underset{\sim}{M} = b + \underset{\sim}{M}$. //

DEFINITION 3 Let $(X,+,\cdot)$ be a ring . And let $\underset{\sim}{H} \in \underset{\sim}{R}(X)$.

The set $X_{\underset{\sim}{H}} = \{x \mid x \in X,\ H(x) = H(0)\}$ is called a base set of $\underset{\sim}{H}$.

PROPOSITION 2 Let $(X,+,\cdot)$ be a ring. And let $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ and $X_{\underset{\sim}{M}}$ is a base set of $\underset{\sim}{M}$. Then $a + \underset{\sim}{M} = b + \underset{\sim}{M}$ iff , for $a,b \in X$,

$$a + X_{\underset{\sim}{M}} = b + X_{\underset{\sim}{M}} .$$

Proof. It is clear that $X_{\underset{\sim}{M}} = M_{\underset{\sim}{M}(0)}$. $X_{\underset{\sim}{M}} \in I(X)$ by Theorem 1, 2) . Let us assume $a + \underset{\sim}{M} = b + \underset{\sim}{M}$. $\underset{\sim}{M}(b - a) = \underset{\sim}{M}(0)$ by Proposition 1. Thus $a - b \in X_{\underset{\sim}{M}}$. Therefore $a + X_{\underset{\sim}{M}} = b + X_{\underset{\sim}{M}}$.

Conversely, let us assume $a + X_{\underset{\sim}{M}} = b + X_{\underset{\sim}{M}}$. Then $b - a \in X_{\underset{\sim}{M}}$. So $\underset{\sim}{M}(b - a) = \underset{\sim}{M}(0)$. Therefore $a + \underset{\sim}{M} = b + \underset{\sim}{M}$. //

COROLLARY 2 Let $(X,+,\cdot)$ be a ring. And let $\underset{\sim}{M} \in \underset{\sim}{I}(X)$. Then $X_{\underset{\sim}{M}} \in I(X)$ and $X - X_{\underset{\sim}{M}} = \{a + X_{\underset{\sim}{M}} \mid a \in X\}$ forms a residue class ring.

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DEFINITION 4 Let $(X,+,\cdot)$ be a ring . And let $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ and $X - \underset{\sim}{M} = \{a + \underset{\sim}{M} \mid a \in X\}$. We define the addition and the product two operations in $X - \underset{\sim}{M}$:

1) $(a + \underset{\sim}{M}) + (b + \underset{\sim}{M}) = (a + b) + \underset{\sim}{M}$.

2) $(a + \underset{\sim}{M})(b + \underset{\sim}{M}) = (ab) + \underset{\sim}{M}$.

The validity of two operations is immediate.

THEOREM 4 Let $(X,+,\cdot)$ be a ring . And let $\underset{\sim}{M} \in \underset{\sim}{I}(X)$, Then

$$X - X_{\underset{\sim}{M}} \cong X - \underset{\sim}{M} \text{ (that is, } X - \underset{\sim}{M} \text{ is also a ring).}$$

Proof. We construct a map $f: X - X_{\underset{\sim}{M}} \to X - \underset{\sim}{M}$, for any $a \in X$,

$$f(a + X_{\underset{\sim}{M}}) = a + \underset{\sim}{M} .$$

f is in-jective by Proposition 2, and for a, b $\in X$,

$$\begin{aligned} f((a + X_{\underset{\sim}{M}}) + (b + X_{\underset{\sim}{M}})) &= f((a + b) + X_{\underset{\sim}{M}}) \\ &= (a + b) + \underset{\sim}{M} \\ &= (a + \underset{\sim}{M}) + (b + \underset{\sim}{M}) \\ &= f(a + X_{\underset{\sim}{M}}) + f(b + X_{\underset{\sim}{M}}); \end{aligned}$$

$$\begin{aligned} f((a + X_{\underset{\sim}{M}})(b + X_{\underset{\sim}{M}})) &= f(ab + X_{\underset{\sim}{M}}) \\ &= ab + \underset{\sim}{M} \end{aligned}$$

$$= (a + \underset{\sim}{M})(b + \underset{\sim}{M})$$

$$= f(a + X_{\underset{\sim}{M}})f(b + X_{\underset{\sim}{M}}).$$

Therefore $X - X_{\underset{\sim}{M}} = X - \underset{\sim}{M}$. //

DEFINITION 5 Let $(X,+,\cdot)$ be a ring . And let $\underset{\sim}{M} \in \underset{\sim}{I}(X)$. Then $X - \underset{\sim}{M}$ is called a residue class ring .

PROPOSITION 3 Let $(X,+,\cdot)$ be a ring . And let $\underset{\sim}{M} \in I(X)$. Then $X \sim X - \underset{\sim}{M}$.

Proof. We con-struct a map $T: X \to X - \underset{\sim}{M}$, for any $a \in X$,

$$T(a) = a + \underset{\sim}{M},$$

it is clear that T is surjective . And for $a,b \in X$,

$$T(a + b) = (a + b) + \underset{\sim}{M} = (a + \underset{\sim}{M}) + (b + \underset{\sim}{M}) = T(a) + T(b);$$

$$T(ab) = ab + \underset{\sim}{M} = (a + \underset{\sim}{M})(b + \underset{\sim}{M}) = T(a)T(b).$$

Therefore $X \sim X - \underset{\sim}{M}$. //

3. Homomorphisms of fuzzy ring

DEFINITION 6 Let f be epimorphism of a ring $(X,+,\cdot)$ into a ring $(X',+,\cdot)$. A fuzzy ideal $\underset{\sim}{M}$ of X is called a kernel of a fuzzy homomorphism of f, if $X_{\underset{\sim}{M}} = \mathrm{Ker} f$.

For example, let f be epimor-phism of a ring $(X,+,\cdot)$ into a ring $(X,+,\cdot)$. A fuzzy subset $\underset{\sim}{M}$ of X is defined as follows:

$$\underset{\sim}{M}(x) = \begin{cases} t, & \text{if } x \in \text{Kerf}, \quad t \in [0,1], \\ 0, & \text{if } x \notin \text{Kerf}. \end{cases}$$

$\underset{\sim}{M}$ is a Kernel of a fuzzy homomorphism of f since the nonempty level subsets of $\underset{\sim}{M}$ have only two X and Kerf. X and Kerf are both ideals of X and $X_{\underset{\sim}{M}} = \text{Kerf}$.

THEOREM 5 Let f be epimorphism of a ring $(X,+,\cdot)$ into a ring $(X',+,\cdot)$. And let $\underset{\sim}{M}$ be a Kernel of a fuzzy homomorphism of f, then

$$X - \underset{\sim}{M} \cong X'.$$

Proof. We have $X - X_{\underset{\sim}{M}} = X - \underset{\sim}{M}$ from Definition 6 and Theorem 4. Since $X - X_{\underset{\sim}{M}} = \underset{\sim}{M} - \text{Kerf} \cong X'$, $X - \underset{\sim}{M} \cong X'$. //

THOEREM 6 Let f be epimor-phism of a ring $(X,+,\cdot)$ into a ring $(X',+,\cdot)$. Then

1) $\underset{\sim}{H} \in \underset{\sim}{R}(X)$ implies $f(\underset{\sim}{H}) \in R(X')$.
2) $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ implies $f(\underset{\sim}{M}) \in \underset{\sim}{I}(X')$.
3) $\underset{\sim}{H}' \in \underset{\sim}{R}(X')$ implies $f^{-1}(\underset{\sim}{H}') \in \underset{\sim}{R}(X)$.
4) $\underset{\sim}{M}' \in \underset{\sim}{I}(X')$ implies $f^{-1}(M') \in \underset{\sim}{I}(X)$.

Proof. 1) Let $\underset{\sim}{H} \in \underset{\sim}{R}(X)$, then $H_t \in R(X)$, $t \in [0,1]$. So $f(H_t) \in R(X')$, $t \in [0,1]$. Therefore $f(\underset{\sim}{H}) = \bigcup_{t \in [0,1]} t\, f(H_t) \in \underset{\sim}{R}(X')$.

2) It can be seen in a similar way that $\underset{\sim}{M} \in \underset{\sim}{I}(X)$ implies $f(\underset{\sim}{M}) \in \underset{\sim}{I}(X')$.

3) Let $\underset{\sim}{H}' \in \underset{\sim}{R}(X')$, then $H'_t \in R(X')$, $t \in [0,1]$. So $f^{-1}(H'_t) \in R(X)$, $t \in [0,1]$ 。 There-fore $f^{-1}(\underset{\sim}{H}') = \bigcup_{t\in[0,1]} t\, f(H'_t) \in \underset{\sim}{R}(X)$.

4) It can be seen in a similar way that $\underset{\sim}{M}' \in \underset{\sim}{I}(X')$ implies $f^{-1}(\underset{\sim}{M}') \in \underset{\sim}{I}(X)$. //

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