HYPERGROUP(I)

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ABSTRACT

The upgrade of all kinds of the structures, such as algebraic structure, ordered structure, topological structure, measurable structure, etc, has been highlighted, with researches in the theoretical basis of fuzzy mathematics. In the paper, we consider the upgrade problem for group.

KEYWARDS: Group, Hypergroup, Generalized quotient group

1. THE GENERAL PROPERTIES OF HYPERGROUP

Let G be a group. In $2^G-\emptyset$ we define a multiplicative opertion: for any A and $B\in 2^G-\emptyset$,

$$AB\triangleq\{ab \mid a\in A,\ b\in B\} \qquad (1.1)$$

It is easy to know the following properties:

(i) $2^G-\emptyset$ is a semigroup with the identity element $\{e\}$ (where e is the identity element in G).

(ii) $A(B\cup C)=AB\cup AC$, $(B\cup C)A=BA\cup CA$

$A(B\cap C)=AB\cap AC$, $(B\cap C)A=BA\cap CA$

(iii) If G is an Abelian group, then for any A and $B\in 2^G-\emptyset$ $AB=BA$.

DEFINITION 1.1 Let $\mathcal{G}\subset 2^G-\emptyset$. $\mathcal{G}$ is called a hypergroup on G, if $\mathcal{G}$ is a group with respect to the multiplicative operation in G. The identity element of $\mathcal{G}$ is denoted by E.

EXAMPLE 1: In the integer additive group (Z,+), we take $E=\{0, 1, 2, \ldots\}$ and $H=\{2m \mid m\in Z\}$. It is clear that $E^2=E$ and H is a subgroup of Z. Put $\mathcal{G}=\{h+E \mid h\in H\}$, it is easy to know that $(\mathcal{G},+)$ is a hypergroup on Z. Now we consider the elements of $\mathcal{G}$. First it is clear that E is the identity element of $\mathcal{G}$ and (-h)+E is the inverse element of h+E. We write

$$A_{-2n}=(-2n)+E\ , \quad A_{2n}=2n+E\ , \quad n=1,2,3,\ldots$$

then

$$\mathcal{G}=\{A_{-2n}\ ,\ E,\ A_{2n}\ ,\ n=1,2,3,\ldots\}$$

and

$$A_{-2n}=\{-2n,-2n+1,\ldots,-1,0,1,2,3,\ldots\}$$

$$A_{2n}=\{2n,2n+1,2n+2,\ldots\}$$

If we define an ordering "$\leqslant$" in $\mathcal{G}$: $A\leqslant B$ iff $A\subset B$, then $(\mathcal{G},\leqslant)$ is a simply ordered set:

$$\ldots A_{2(n+1)}<A_{2n}<\ldots<E<\ldots<A_{-2n}<A_{-2(n+1)}<\ldots$$

For any $A,B,C\in\mathcal{G}$, if $A\leqslant B$, then it is easy to know that $AC\leqslant BC$ and $CA\leqslant CB$. Hence $(\mathcal{G},\leqslant)$ is a simply ordered group. NOw we take the mapping:

$$f:\ H\longrightarrow\mathcal{G}$$
$$h\longmapsto h+E$$

It is easy to prove that f is a isotone isomorphism from $(H,\leqslant)$ to $(\mathcal{G},\leqslant)$.

PROPOSITION 1.1 Let $\mathcal{G}$ be a hypergroup and $\leqslant\ =\subset$, the $\mathcal{G}$ is a partially ordered group.

PROOF. For any $A,B,C\in\mathcal{G}$, if $A\leqslant B$, then $AC\cup BC=(A\cup B)C=BC$, thus $AC\leqslant BC$. In like manner we have $CA\leqslant CB$. This means that $\mathcal{G}$ is a partially ordered group. Q.E.D.

PROPOSITION 1.2 E is a semigroup with respect to the multiplication in G.

PROOF. We only need to note $E^2=E$. Q.E.D.

PROPOSITION 1.3 If $\mathcal{G}$ is a hypergroup, then

$$(\forall A\in \mathcal{G})(\mathrm{card}A=\mathrm{card}E) \qquad (1.2)$$

PROOF. In one respect we have

$AE=A$ implies $(\forall a\in A)(aE\subset A)$ implies $\mathrm{card}E=\mathrm{card}(aE)\leqslant \mathrm{card}A$

In the other respect we have

$A^{-1}A=E$ implies $(\forall b\in A^{-1})(bA\subset E)$ implies $\mathrm{card}A=\mathrm{card}(bA)\leqslant \mathrm{card}E$

Hence card=cardE. Q.E.D.

PROPOSITION 1.4 If $\mathcal{G}$ is a hypergroup, then

$$(\forall A,B\in\mathcal{G})(A\cap B\neq\emptyset \text{ implies } \mathrm{card}(A\cap B)=\mathrm{card}E) \qquad (1.3)$$

PROOF. In one respect we have ($c\in A\cap B$ implies $cE\subset A$ and $cE\subset B$ implies $cE\subset A\cap B$ implies $\mathrm{card}E=\mathrm{card}(cE)\leqslant \mathrm{card}(A\cap B)$).
In the other respect it is caear that $\mathrm{card}(A\cap B)\leqslant \mathrm{card}E$.
Hence $\mathrm{card}(A\cap B)=\mathrm{card}E$. Q.E,D.

THEOREM 1.1 Let $\mathcal{G}$ be a hypergroup. If E is a subgroup of G, then

$$G^*\triangleq \cup\{A \mid A\in\mathcal{G}\} \qquad (1.4)$$

is also a subgroup of G, and

$$\mathcal{G}=G^*/E \qquad (1.5)$$

PROOF. (i) For any $A\in\mathcal{G}$ and any $a\in A$, we have $Ea\subset A$ by $EA=A$. It can be proved that $Ea=A$. If it is false, then $\exists\, b\in A$ such that $b\notin Ea$. We can prove $ab^{-1}\notin E$. If this is also false, then $\exists c\in E$ such that $ab^{-1}=c$, thus $b=c^{-1}a\in Ea$, this is in contradication to $b\notin Ea$, hence $ab^{-1}\notin E$. Now we take $d\in A^{-1}$, clearly $(ad)(bd)^{-1}\in E$, by this we have $ab^{-1}\in E$, this is in contradication to $ab^{-1}\notin E$. Hence we have $Ea=A$. In like manner we can prove $aE=A$, thus $aE=Ea$.
From this we have

$$(\forall a\in G^*)(aE=Ea) \qquad (1.6)$$

$$\mathcal{G}=\{aE \mid a\in G^*\} \qquad (1.7)$$

(ii) Prove that G^* is a subgroup of G.
In one respect, for any $a,b\in G^*$, $\exists A,B\in\mathcal{G}$ such that $a\in A$ and $b\in B$. Since $\mathcal{G}$ is a group, $\exists C\in\mathcal{G}$ such that $AB=C$. Thus $\exists\, c\in C$ such that $ab=c$, hence $ab\in C\subset G^*$. This means that G^* is closed with respect to the multiplicative operation in G.
In other respect, for any $a\in G^*$, $\exists A\in\mathcal{G}$ such that $a\in A$. From $AA^{-1}=E$ and $e\in E$ $\exists b\in A$ and $b'\in A^{-1}$ such that $bb'=e$. Thus $b^{-1}=b'\in A^{-1}$. Since $A=bE$ $\exists\, c\in E$ such that $a=bc$. Hence $a^{-1}=c^{-1}b^{-1}\in Eb^{-1}=A^{-1}\subset G^*$. This means

that G^* is closed with respect to a^{-1} for any $a\in G^*$.
From the two respects we know that G^* is a subgroup of G.
(iii) From (i) and (ii) E is just a normal subgroup of G^*, hence $\mathcal{G}=G^*/E$. Q.E.D.

COROLLARY 1 If E which is the identity element is a subgroup of G, then $\mathcal{G}$ is a quotoent group iff $G=G^*$.

COROLLARY 2 Let $\mathcal{G}$ be a hypergroup. If the elements in E are all finite order, then $\mathcal{G}=G^*/E$.

COROLLARY 3 If G is a periodic group, then $\mathcal{G}=G^*/E$.

COROLLARY 4 If E which is the identity element is a finite set, then $\mathcal{G}=G^*/E$.

PROOF. By means of $E^n=E$, $(n=1,2,\ldots)$, it is easy to prove that the elements in E are all finite order, hence $\mathcal{G}=G^*/E$. Q.E.D.

COROLLARY 5 If G is a finite group, the $\mathcal{G}=G^*/E$.

THEOREM 1.2 Let $\mathcal{G}$ be a hypergroup on a group G and G' be a group. If $f:G\longrightarrow G'$ is a homomorphism, then

$$\mathcal{G}'\triangleq\{f(A)\mid A\in\mathcal{G}\} \tag{1.8}$$

is a hypergroup on G', and $\mathcal{G}\sim\mathcal{G}'$.

The proof is simple, it is omitted.

THEOREM 1.3 Let $\mathcal{G}'$ be a hypergroup on a group G' and G be a group. If $f:G\longrightarrow G'$ is a homomorphism, then

$$\mathcal{G}\triangleq\{f^{-1}(A')\mid A'\in G'\} \tag{1.9}$$

is a hypergroup on G, and $\mathcal{G}\sim\mathcal{G}'$.

The proof is simple, it is omitted.

THEOREM 1.4 Let $\mathcal{G}$ be a hypergroup, $B\in 2^G-\emptyset$ and $B^2=B$. If $AB=BA$ for any $A\in\mathcal{G}$, then

$$\mathcal{G}_B\triangleq\{AB\mid A\in\mathcal{G}\} \tag{1.10}$$

is also a hypergroup on G, and $\mathcal{G}\sim\mathcal{G}_B$.

PROOF. It is easy to prove that the mapping

$$f:\ \mathcal{G}\longrightarrow\mathcal{G}_B$$
$$A\longmapsto AB$$

[illegible] surjective homomorphism, hence $\mathcal{G}_B$ is also a hypergroup on G. Q.E.D.

COROLLARY Let $\mathcal{G}$ be a hypergroup. If N is a normal subgroup of G, then $\mathcal{G}_N \triangleq \{ AN \mid A\in\mathcal{G}\}$ is also a hypergroup on G, and $\mathcal{G}\sim\mathcal{G}_N$.

2. GENERALIZED QUOTIENT GROUP

PROPOSITION 2.1 Let $E\in 2^G-\emptyset$. If $e\in E$, then E is a subgroup of G iff $E^2=E$.

The proof is simple, it is omitted.

THEOREM 2.1 Let $E\in 2^G-\emptyset$, $E^2=E$ and H be a subgroup of G. If $(\forall x\in H)(xE=Ex)$, then

$$\mathcal{G}\triangleq\{xE \mid x\in H\} \tag{2.1}$$

is a hypergroup on G, and $H\sim\mathcal{G}$.

PROOF. It is easy to prove that the mapping

$$f: H \longrightarrow \mathcal{G}$$
$$x \longmapsto xE$$

is a surjective homomorphism, hence $\mathcal{G}$ is a hypergroup on G. Q.E.D.

DEFINITION 2.1 Let E be a subsemigroup of G and $e\in E$. E is called a normal subsemigroup of G, if $(\forall x\in G)(xE=Ex)$.

DEFINITION 2.2 Let E be a normal subsemigroup of G. Write

$$G|E=\{xE \mid x\in G\} \tag{2.2}$$

From the theorem 2.1 we know that $G|E$ is a hypergroup on G. $G|E$ is called a generalized quotient group on G.

EXAMPLE 2: In the example 1 if we take H=Z, then $\mathcal{G}$ is a generalized quotient group on Z.

NOTE: Let $E\in 2^G-\emptyset$. That $e\in E$ is not a necessary condition which is that $E^2=E$. For example, if E is the set of all positive rational numbers in the rational number group (Q,+), then $E^2=E$ but $0\notin E$.

PROPOSITION 2.2 If E is a normal subsemigroup, then

$$K\triangleq\{x\in E \mid x^{-1}\in E\} \tag{2.3}$$

is a normal subgroup of G.

The K is called Kernel of E.

THEOREM 2.2 Let $G|E$ be a generalized quotient group and K be the kernel of E, then the mapping

$$f: G \longrightarrow G|E$$
$$x \longrightarrow xE$$

is a surjective homomorphism and kerf=E, thus $G/K \cong G|E$. Especially, $G|E=G/K$ when K=E, and $G \cong G|E$ when $K=\{e\}$.

PROOF. Clearly f is a surjective homomorphism. Now we prove that kerf=K. For any $x\in K$ clearly $xE \subset E$. If $b\in E$ then $x^{-1}b\in E$, thus $b=x(x^{-1}b)\in xE$, thus xE=E, hence $x\in kerf$, i.e. $K\subset kerf$. If $x\in kerf$, then $x=xe\in E$ by xE=E. For $e\in E$, $\exists b\in E$ such that xb=e, thus $b=x^{-1}\in E$, hence $x\in K$, i.e. $kerf\subset K$. These mean that kerf=K, hence $G/K \cong G|E$.
Q.E.D.

Let G|E be a generalized quotient group. For any $x\in G$ the mapping

$$\phi_x: G|E \longrightarrow G|E$$
$$A \longmapsto xAx^{-1} \qquad (2.4)$$

is an automorphism on G|E. It is called a induced automorphism on G|E, and the set of all induced automorphism on G|E is denoted by I(G|E). Clearly I(G|E) is a group with respect to the operation for composite mapping. It is called induced automorphism group.

Let K be a normal subgroup of G. $x\in G$ is called a commutative extension element, if

$$(\forall a\in G)(\exists m\in K)(xax^{-1}=am) \qquad (2.5)$$

The set of all commutative extension elements of K is denoted by L(K), that is called commutative extensioner.

It is easy to know that $K\subset L(K)$.

THEOREM 2.3 Let G|E be a generalized quotient group. If K is the kernel of E, then

$$I(G|E) \cong G/L(K) \qquad (2.6)$$

Especially, if L(K)=G then I(G|E) only contains the identity automorphism and if L(K)=K then

$$I(G|E) \cong G/K \cong G/E \qquad (2.7)$$

PROOF. Firstly it is easy to know that the mapping

$$f: G \longrightarrow I(G|E)$$
$$x \longmapsto \phi_x$$

is a surjective homomorphism. Secondly, for any $A=aE\in G|E$ we have

$$xAx^{-1}=A \text{ iff } (\exists m\in K)(xax^{-1}=am)$$

This means that $xAx^{-1}=A$ iff $x\in L(K)$, thus $\ker f=L(K)$, hence $L(K)$ is a normal subgroup of G and $G/L(K) \cong I(G\ E)$. Q.E.D.

THEOREM 2.4 Let $\mathcal{G}$ be a hypergroup. If $e\in E$ then $\mathcal{G}$ is a generalized quotient group on G^*, i.e. $\mathcal{G}=G^*|E$.

PROOF. (i) Clear E is a subsemigroup with the identity element from the proposition 2.1.

(ii) $\forall A\in\mathcal{G}$, $\exists a\in A, \exists b\in A^{-1}$ by $AA^{-1}=E$ and $e\in E$, such that $ab=e$. It can be proved that $A=aE$. Firstly that $aE\subset A$ is clear. Secondly, $\forall c\in A$, $c=ec=(ab)c=a(bc)$ and $bc\in A^{-1}A=E$, thus $c\in aE$, hence $A\subset aE$. This proves that $A=aE$. In like manner we can prove that $A=Ea$, thus $aE=Ea$, and we have

$$\mathcal{G}=\{aE \mid a\in G^*\} \quad (2.8)$$

(iii) It can be proved that G^* is a subgroup of G like the proof to the theorem 1.1. Hence $\mathcal{G}=G^*|E$. Q.E.D.

EXAMPLE 3: Let G be a cyclic group and $\mathcal{G}$ be a hypergroup on G. Clearly $e\in E$, thus $\mathcal{G}$ must be a generalized quotient group on G^*. Since G^* is a subgroup of G, G^* is also a cyclic group. When G^* is a finite order cyclic group, $\mathcal{G}$ is a qoutinet group of G^*, thus $\mathcal{G}$ is also a finite order cyclic group.
When G^* is a infinite cyclic group, we have the following two cases:

(i) If E is a subgroup of G, the $\mathcal{G}=G^*/E$, thus $\mathcal{G}$ is a finite order cyclic group.

(ii) If E is a subsemigroup of G, then $K=\{e\}$, where K is the kernel of E, and $\mathcal{G}\cong G^*$, thus $\mathcal{G}$ is a infinite order cyclic group.

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