LATTICEIZATION GROUP (I)

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The concept of the hypergroup was firstly advanced in the paper (1), where it is discussed in detail and some interesting results have been obtained. In the paper, we consider the hypergroup from the point of view on lattice theory, and also firstly advance the concept of latticeization group, which will show the structure of the hypergroup in the extensive cases.

KEYWARDS: Group, Lattice, Latticeization Group, Latticeization Quotient Group, Latticeization Weak Quotient Group.

1. INTRODUCTION

In recent years, the upgrade of all kinds of mathematical structures from their universes to their power sets has been brought to extensive attention. Especially, with the researches on the theoretical basis of fuzzy mathematics, the problem of the upgrade to the structures are more and more highlighted, such as the upgrade of ordered structure, topological structure (or more extensive latticeization topology), measurable structure, etc. It is easy to imagine that the upgrade of algebraic structure will also be interesting. The paper (1) firstly broke through the frame of quotient group, and the concept of the hypergroup was advanced in there, which regards quotient group as a special case, and some interesting results have been obtained. In the paper, the point of view is risen once more, and we consider group from the viewpoint of lattice theory, which differs from the known lattice group or ordered group but is a new structure which will regards the hypergroup and fuzzy hypergroup as special cases.

2. THE DEFINITION OF LATTICEIZATION GROUP AND ITS LOCAL PROPERTIES

Let $G=(G,\cdot,\vee,\wedge)$ be both a groupoid and a complete distributive lattice, in which " $\geqslant$ " is the corresponding ordering relation, 0 and 1 is respectively the least element and the greatest element in G. But G may not be a lattice groupoid. We stipulate that the elements in G are respectively denoted by a,b,c,... and $\alpha,\beta,\gamma,\ldots$, when G is respectively regarded as a groupoid and a lattice.

For any $\alpha\in G$, write

$$\underset{\cdot}{\alpha} \triangleq \{\beta \mid \beta\in G, \alpha\geqslant\beta\}$$

DEFINITION 2.1 In G a new algebraic operation "o" is introdeced by

$$\alpha\circ\beta \triangleq \vee\{a\cdot b \mid a\in\underset{\cdot}{\alpha}, b\in\underset{\cdot}{\beta}\} \qquad (2.1)$$

(G,o) is called a latticeization groupoid.

PROPOSITION 2.1 (i) (G,o) is a lattice groupoid, i.e., (G,o) is a groupoid with

$$(\forall\alpha,\beta,\gamma\in G)(\alpha\geqslant\beta \text{ implies } (\alpha\circ\gamma\geqslant\beta\circ\gamma\ ,\gamma\circ\alpha\geqslant\gamma\circ\beta)) \qquad (2.2)$$

(ii) That $(G,\cdot)$ is commutative implies that (G,o) is commutative.

(iii) That $(G,\cdot)$ is a semigroup implies that (G,o) is a semigroup. ※

NOTE 1: Clearly, (2.2) is true iff the following (2.3) is true.

$$\left.\begin{aligned}(\alpha\vee\beta)\circ\gamma=\alpha\circ\gamma\vee\beta\circ\gamma\ , &\quad \gamma\circ(\alpha\vee\beta)=\gamma\circ\alpha\vee\gamma\circ\beta\\ (\alpha\wedge\beta)\circ\gamma=\alpha\circ\gamma\wedge\beta\circ\gamma\ , &\quad \gamma\circ(\alpha\wedge\beta)=\gamma\circ\alpha\wedge\gamma\circ\beta\end{aligned}\right\} \qquad (2.3)$$

NOTE 2: (G,o) may not be a group even if $(G,\cdot)$ is a group.

NOTE 3: If $a\in G$ and $\beta\in G$ then $a\circ\beta$ is stipulated by

$$a\circ\beta \triangleq \vee\{a\cdot b \mid b\in\underset{\cdot}{\beta}\} \qquad (2.4)$$

Now we always suppose that $(G,\cdot)$ is a group where e is the identity element in G.

DEFINITION 2.2 Let $L \subset G$. If (L, o) is a group then L is called a latticeization group on G, where the identity element in L is denoted by ε.

PROPOSITION 2.2 $(\varepsilon, \cdot)$ is a subsemigroup of $(G, \cdot)$.

PROOF. We only need to note $\varepsilon o \varepsilon = \varepsilon$. #

PROPOSITION 2.3 If L is a latticeization group then

$$(\forall \alpha \in L)(\mathrm{card}\, \alpha = \mathrm{card}\, \varepsilon) \tag{2.5}$$

PROOF. From $\alpha o \varepsilon = \alpha$ we have $\alpha o \varepsilon = \alpha$. Take $a \in \alpha$, then $a o \varepsilon \subset \alpha$, so

$$\mathrm{card}\, \varepsilon \leq \mathrm{card}(a o \varepsilon) \leq \mathrm{card}\, \alpha$$

From $\alpha^{-1} o \alpha = \varepsilon$ we have $\alpha^{-1} o \alpha = \varepsilon$. Take $b \in \alpha^{-1}$, then $b o \alpha \subset \varepsilon$, so

$$\mathrm{card}\, \alpha \leq \mathrm{card}(b o \alpha) \leq \mathrm{card}\, \varepsilon$$

Hence $\mathrm{card}\, \alpha = \mathrm{card}\, \varepsilon$. #

PROPOSITION 2.4 If L is a latticeization group on G, then

$$(\forall \alpha, \beta \in L)(\mathrm{card}(\alpha \wedge \beta) = \mathrm{card}\, \varepsilon) \tag{2.6}$$

PROOF. In one respect we have ($d \in \alpha \wedge \beta$ implies $(d o \varepsilon \subset \alpha, d o \varepsilon \subset \beta)$ implies $d o \varepsilon \subset \alpha \cap \beta = \alpha \wedge \beta$ implies $\mathrm{card}\, \varepsilon \leq \mathrm{card}(d o \varepsilon) \leq \mathrm{card}(\alpha \wedge \beta)$). In the other respect it is clear that $\mathrm{card}(\alpha \wedge \beta) \leq \mathrm{card}\, \varepsilon$. Hence $\mathrm{card}(\alpha \wedge \beta) = \mathrm{card}\, \varepsilon$. #

3. LATTICEIZATION QUOTIENT GROUP

DEFINITION 3.1 Let $\eta \in G$. η is called a regular element if η is a subgroup of G; η is called a normal element if η is a regular element and satisfies the normal condition:

$$(\forall a \in G)(\ a o \eta = \eta o a) \tag{3.1}$$

η is called a weak regular element if η is a subsemigroup with the identity element e; η is called a weak normal element if η is a weak regular element with the normal condition.

It is easy to prove

PROPOSITION 3.1 Let $\eta \in G$. If $(\eta, \cdot)$ is a normal subgroup of G, then η is a normal element. Conversely it may not be true.

DEFINITION 3.2 Let η be a regular element, $a \in G$. $a o \eta$ is called a left coelement and $\eta o a$ a right coelement. $a o \eta$ is called a coelement if $a o \eta = \eta o a$.

THEOREM 3.1 If η is a normal element, then

$$G/\eta \triangleq \{ a o \eta \mid a \in G \} \qquad (3.2)$$

is a latticeization group on G and $G \sim G/\eta$.

PROOF. It is easy to see that the mapping

$$f : G \longrightarrow G/\eta$$
$$a \longmapsto a o \eta$$

is a surjective homomorphism, thus G/η is a latticeization group on G and $G \sim G/\eta$. #

DEFINITION 3.3 If η is a normal element, then the latticeization group $(G/\eta, o)$ is called a latticeization quotient group of G.

4. THE STRUCTURE OF LATTICEIZATION GROUP

Let L be a latticeization group, write

$$G^{*} \triangleq \cup \{ \underset{\cdot}{\alpha} \mid \alpha \in L \} \qquad (4.1)$$

THEOREM 4.1 Let L be a latticeization group on G. If ε is a regular element, then

(i) G^{*} is a subgroup of G.

(ii) ε is a normal element of G^{*}.

(iii) $L = G^{*}/\varepsilon$.

PROOF. (a) For any $\alpha \in L$ and any $a \in \underset{\cdot}{\alpha}$, we have $\varepsilon o a \leq \alpha$ by $\varepsilon o \alpha = \alpha$. It can be proved that $\varepsilon o a = \alpha$. This only needs to prove that $\underset{\cdot}{\varepsilon o a} = \underset{\cdot}{\alpha}$. If it is false, then $\exists b \in \underset{\cdot}{\alpha}$ by $\underset{\cdot}{\varepsilon o a} \subset \underset{\cdot}{\alpha}$ such that $b \notin \underset{\cdot}{\varepsilon o a}$. We can prove $a \cdot b^{-1} \notin \underset{\cdot}{\varepsilon}$. If this also false, then $\exists c \in \underset{\cdot}{\varepsilon}$ such that $a \cdot b^{-1} = c$, thus $b = c^{-1} \cdot a \in \underset{\cdot}{\varepsilon o a}$, this is in contradication to $b \notin \underset{\cdot}{\varepsilon o a}$. Now we take $d \in \underset{\cdot}{\alpha}^{-1}$, clearly $(a \cdot d) \cdot (b \cdot d)^{-1} \in \underset{\cdot}{\varepsilon}$, by this we have $a \cdot b^{-1} \in \underset{\cdot}{\varepsilon}$, this is in contradication to $a \cdot b^{-1} \notin \underset{\cdot}{\varepsilon}$. Hence $\underset{\cdot}{\varepsilon o a} = \underset{\cdot}{\alpha}$, i.e., $\varepsilon o a = \alpha$. In like manner we can prove $a o \varepsilon = \alpha$. Thus $a o \varepsilon = \varepsilon o a$, so we have

$$(\forall a \in G^{*})(a o \varepsilon = \varepsilon o a)$$
$$L = \{ a o \varepsilon \mid a \in G^{*} \}$$

(b) Now we prove that G^{*} is a subgroup of G.

In one respect, for any $a, b \in G^{*}$, $\exists \alpha, \beta \in L$ such that $a \in \underset{\cdot}{\alpha}$, $b \in \underset{\cdot}{\beta}$. Since L is a group, $\exists \gamma \in L$ such that $\alpha o \beta = \gamma$. Thus $\exists c \in \underset{\cdot}{\gamma}$ such that

$a\cdot b=c$, hence $a\cdot b\in\gamma < G^*$. This means that $(G^*,\cdot)$ is a groupoid. In other respect, for any $a\in G^*$, $\exists\alpha\in L$, such that $a\in\alpha$. From $\alpha\circ\alpha^{-1}=\varepsilon$ and $e\in\varepsilon$, $\exists b\in\alpha$ and $b'\in\alpha^{-1}$ such that $b\cdot b'=e$, thus $b^{-1}=b'\in\alpha^{-1}$. Since $\alpha=b\circ\varepsilon$, $\exists c\in\varepsilon$ such that $a=b\cdot c$, hence $a^{-1}=c^{-1}\cdot b^{-1}\in\varepsilon\circ b^{-1}=\alpha^{-1} < G^*$. This means that G^* is closed with respect to a^{-1} for any $a\in G^*$. From the two respects we know that G^* is a subgroup of G.
(c) From (a) and (b) ε is just a normal element of G^*. Hence $L=G^*/\varepsilon$. #

COROLLARY 1 If ε is a regular element of G, then $L=G/\varepsilon$ iff $G=G^*$. #

COROLLARY 2 If the elements in ε are all finite order, then $L=G^*/\varepsilon$. #

COROLLARY 3 If ε is a periodic group, then $L=G^*/\varepsilon$. #

COROLLARY 4 If ε is a finite set, then $L=G^*/\varepsilon$. #

COROLLARY 5 If G is a finite group, then $L=G^*/\varepsilon$. #

COROLLARY 6 Let ε is a regular element of G. If $1\in L$, then $L=G/\varepsilon$. #

5. HOMOMORPHISM AND ISOMORPHISM OF LATTICEIZATION GROUP

DEFINITION 5.1 Let L_1 and L_2 be two complete distributive lattices. From a mapping $f: L_1 \longrightarrow L_2$ we make a mapping:

$$f_{-1}: L_2 \longrightarrow L_1$$
$$\beta \longmapsto f_{-1}(\beta) \triangleq \vee\{\alpha\in L_1 \mid f(\alpha)\in\beta\}$$

f is called a lattice mapping from L_1 to L_2, if

(i) $f(0)=0$
(ii) $(\forall A\subset L_1)(f(\vee\{\alpha\mid\alpha\in A\})=\vee\{f(\alpha)\mid\alpha\in A\})$
(iii) $(\forall B\subset L_2)(f_{-1}(\vee\{\beta\mid\beta\in B\})=\vee\{f_{-1}(\beta)\mid\beta\in B\})$

where f_{-1} is called the lattice inverse mapping of f.

PROPOSITION 5.1 (i) f and f_{-1} are all order-preserving mapping.
(ii) $f_{-1}(f(\alpha))\geq\alpha$, and when f is a injection $f_{-1}(f(\alpha))=\alpha$.

(iii) $f(f_{-1}(\beta))\leqslant\beta$, and when f is a surjection $f(f_{-1}(\beta))=\beta$. #

PROPOSITION 5.2 Let $f: L_1 \longrightarrow L_2$ and $g: L_2 \longrightarrow L_3$ be all lattice mappings, then $gof: L_1 \longrightarrow L_3$ is also a lattice mapping and $(gof)_{-1}=f_{-1}og_{-1}$. #

PROPOSITION 5.3 Let $f: L_1 \longrightarrow L_2$ be a lattice mapping. If f is bijection, then f_{-1} is also a lattice mapping (clearly it is a bijection) and $(f_{-1})_{-1}=f$, $f_{-1}of=I_{L_1}$, $fof_{-1}=I_{L_2}$. #

DEFINITION 5.2 Let G_1 and G_2 be all both groups and complete distributive lattices, $f: G_1 \longrightarrow G_2$ be a lattice mapping. If f is a isomorphism (homomorphism) of groups, then f is called a latticeization isomorphism (homomorphism).

THEOREM 5.1 Let L be a latticeization group, $f: G \longrightarrow G_1$ be a latticeization homomorphism. If $f|_L$ is a injection, then

$$L_1 \triangleq \{f(\alpha) \mid \alpha\in L\}$$

is a latticeization group on G_1 and $(L,o)\cong(L_1,o)$.

PROOF. Let $g=f|_L$, then g is a bijection from L to L_1, so $\alpha\geqslant\beta$ iff $g(\alpha)\geqslant g(\beta)$. Hence

$$\begin{aligned} g(\alpha o\beta) &= f(\alpha o\beta) \\ &= f(\vee\{a\cdot b \mid a\leqslant\alpha, b\leqslant\beta\}) \\ &= \vee\{f(a)\cdot f(b) \mid f(a)\leqslant f(\alpha), f(b)\leqslant f(\beta)\} \\ &= f(\alpha)of(\beta) \\ &= g(\alpha)og(\beta) \end{aligned}$$

This means that g is a isomorphism from (L,o) to (L_1,o). #

THEOREM 5.2 Let L_1 be a latticeization group on G_1, $f:G\rightarrow G_1$ be a latticeization surjective homomorphism. If $f^{-1}|_{L_1}$ is a injection, then

$$L\triangleq\{f^{-1}(\beta) \mid \beta\in L_1\}$$

is a latticeization group on G and $(L,o)\cong(L_1,o)$.

PROOF. Let $h=f^{-1}|_{L_1}$, then h is a bijection from L_1 to L and it is easy to see that h is also union-preserving. Thus, just like the proof of the theorem 5.1, we can prove that $(L,o)\cong(L_1,o)$. #

THEOREM 5.3 Let L_1 be a latticeization group $f:G \longrightarrow G_1$ be a surjective homomorphism. If $f_{-1}|_{L_1}$ is a injection, then

$$L=\{f_{-1}(\beta)\mid \beta\in L_1\}$$

is a latticeization group on G and $(L,o)\cong(L_1,o)$.

The proof is like the proof of the theorem 5.2, here is omitted.

THEOREM 5.4 Let L be a latticeization group on G, $\eta\in G$ with $\eta o\eta=\eta$. If η satisfies $(\forall\alpha\in L)(\alpha o\eta=\eta o\alpha)$, then

$$L_\eta \triangleq \{\alpha o\eta \mid \alpha\in L\}$$

is a latticeization group on G and $L\sim L_\eta$.

PROOF. It is easy to see that the mapping

$$f: L \longrightarrow L_\eta$$
$$\alpha \longmapsto \alpha o\eta$$

is a surjective homomorphism, thus L_η is a latticeization group on G. #

COROLLARY Let L be a latticeization group on G. If η is a normal element, then $L_\eta\triangleq\{\alpha o\eta \mid \alpha\in L\}$ is a latticeization group on G and $L\sim L_\eta$. #

THEOREM 5.5 Let β be a regular element, $\eta\in G$ with $\eta o\eta=\eta$. If it satisfies $(\forall a\in\beta)(ao\eta=\eta oa)$, then

$$L\triangleq\{ao\eta \mid a\in\beta\}$$

is a latticeization group on G and $\beta\sim L$.

PROOF. Clearly the mapping

$$f:\beta \longrightarrow L$$
$$a \longmapsto ao\eta$$

is a bijection, and we have

$$f(a\cdot b)=(a\cdot b)o\eta=(aob)o\eta o\eta=(ao\eta)o(bo\eta)=f(a)of(b)$$

Hence $(\beta,\cdot)\sim(L,o)$. #

6. LATTICEIZATION WEAK QUOTIENT GROUP

PROPOSITION 6.1 Let $\eta\in G$. If $e\in\eta$, then η is a subsemigroup of G iff $\eta o\eta=\eta$.

PROOF. " $\Longrightarrow$ ": $\eta o\eta\subset\eta$ since η is a subsemigroup of G. Besides, clearly $\eta o\eta\supset eo\eta=\eta$, so $\eta o\eta=\eta$.

"$\Longleftarrow$": Clear #

DEFINITION 6.1 Let η be a weak normal element, write

$$G|\eta \triangleq \{ a\circ\eta \mid a\in G \}$$

It is easy to see $G|\eta$ is a latticeization group on G. $G|\eta$ is called a latticeization weak quotient group.

THEOREM 6.1 Let L be a latticeization group on G. If $e\in\dot{\varepsilon}$, then $L=G^*|\varepsilon$.

PROOF. For any $\alpha\in L$, by $\alpha\circ\alpha^{-1}=\varepsilon$ and $e\in\dot{\varepsilon}$, $\exists a\in\dot{\alpha}$ and $b\in\dot{\alpha}^{-1}$ such that $a\cdot b=e$. It can be proved that $\alpha=a\circ\varepsilon$. That $a\circ\varepsilon\subset\alpha$ is clear. Besides, $\forall$ $c\in\dot{\alpha}$, $c=e\cdot c=(a\cdot b)\cdot c=a\cdot(b\cdot c)$, but $b\cdot c\in\alpha^{-1}\circ\alpha=\dot{\varepsilon}$, so $c\in a\circ\dot{\varepsilon}$, thus $\dot{\alpha}\subset a\circ\dot{\varepsilon}$. Hence $\alpha=a\circ\varepsilon$. In the same way we can prove that $\alpha=\varepsilon\circ a$, thus $a\circ\varepsilon=\varepsilon\circ a$ and $L=\{ a\circ\varepsilon \mid a\in G^* \}$.
In addition, like the proof of the theorem 4.1 we know that G^* is a subgroup of G, so ε is a weak normal element of G^*, thus $L=G^*|\varepsilon$. #

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