ON MEASURING CONTROLLABILITY PROPERTY FOR SYSTEMS DESCRIBED BY FUZZY RELATION EQUATIONS

Siegfried Gottwald
Logic Group, Dept. of Philosophy
Karl-Marx University, Leipzig
GDR

Witold Pedrycz
Dept. of Automatic Control
and Computer Science Silesian
Technical Univ. Gliwice, and
Research and Development Center
of Electronic Medical Instr.
Zabrze, Poland

Abstract The paper is devoted to discussing the problem of measuring the controllability property for systems for which the proposed models are fuzzy ones. A non-additive measure, a so-called fuzzy measure is considered as a tool for expressing the quality of the model, while a fuzzy integral is sought as an appropriate means for numerical quantification of the property of controllability.

Introduction

It is an obvious fact that the more adequate a model of any system, the more reliable knowledge it represents concerning properties of the system. We shall focus our attention on a wide class of models, describing systems with a factor of fuzziness and consider the controllability property of the system. Loosely speaking, this feature expresses to which extent one is able to attain a goal from an actual state of the system applying a specified control.

A question of genuine importance while speaking about applicability of the constructed model is how the property of controllability of the system, not the relevant model, may be evaluated. The fuzzy integral /7//8/ will be of interest to express the above property in a quantitative fashion. Without any significant loss of generality of our deliberations, we shall discuss the class of models described by fuzzy relation equations cf. /3/, /4/, /5/.

$$B = A \circ R \tag{1}$$

where A is viewed as a fuzzy input/control/, and B is a fuzzy output/state/ defined in finite universes of discourse, namely $\mathbb{A}=\{a_1, a_2, \ldots, a_n\}$, $\mathbb{B}=\{b_1, b_2, \ldots, b_m\}$. Relationships between the input and output are given by a fuzzy relation R. With "$\circ$" for max-t composition (1) is read as,

$$B(b_j) = \max_{1 \le i \le n} [A(a_i) \, t \, R(a_i, b_j)] \tag{2}$$

$b_j \in \mathbb{B}$, where "t" means any t-norm cf. e.g. /6/. In Section 2 we discuss the problem of the calculation of the fuzzy relation R, and the fuzzy measure associated with the model. while in sequel/Section 3/ we concern ourselves with measuring the controllability property by computation of the fuzzy integral.

Calculating the relation of the model and the fuzzy measure

In order to determine the fuzzy relation R of the model, let us take into account a finite collection of input-output fuzzy data,

$$\begin{matrix} A^{(1)} & B^{(1)} \\ A^{(2)} & B^{(2)} \\ \vdots & \\ A^{(N)} & B^{(N)} \end{matrix} \qquad (3)$$

From the theory of fuzzy relation equations we have /4/

Proposition 1. If $\mathcal{R}=\bigcap_{l=1}^{N}\{R:\mathbb{A}\times\mathbb{B}\rightarrow[0,1] \mid A^{(l)}\Box R=B^{(l)}\}\neq\emptyset$ then $\hat{R}=\bigcap_{l=1}^{N}(A^{(l)}\varphi B^{(l)})$ is the greatest element of $\mathcal{R}$, $\hat{R}=\max\mathcal{R}$.

The main difficulty we have to underline, is the fact that $\mathcal{R}$ may be empty. But, even knowing $\mathcal{R}$ to be empty, let us calculate the fuzzy relation of our model as intersection of all $A^{(l)}\varphi B^{(l)}$, $l=1,2,\dots,N$. A straightforward consequence of the use of this approach is possibly a lack of equality of the fuzzy sets $B^{(l)}$ and the corresponding ones coming from the equation of the model. The equality of the model now may be measured in output space $\mathbb{B}$; hence we ask how "far" $\hat{B}^{(k)}=A^{(k)}\Box\hat{R}$ is from the fuzzy set $B^{(k)}$. For this purpose we slightly modify a generalized definition of equality of any two fuzzy sets defined in the same space, cf. /2/. We introduce a fuzzy set $\underline{\gamma}:\mathbb{B}\rightarrow[0,1]$ defined pointwise,

$$\underline{\gamma}=[\gamma_1\ \gamma_2\dots\gamma_m] \qquad \gamma_i=\underline{\gamma}(b_i) \qquad (4)$$

where each coordinate of $\underline{\gamma}$, for instance γ_j is viewed as a degree of equality of all pairs of fuzzy sets $\hat{B}^{(k)}$ and $B^{(k)}$ at the point $b_j\in\mathbb{B}$,

$$\gamma_j=\bigwedge_{k=1}^{N}[\hat{B}^{(k)}(b_j)\leftrightarrow B^{(k)}(b_j)] \qquad (5)$$

Here "$\leftrightarrow$" denotes a generalized biimplication operator as e.g. used in /1//2/. Thus we have

$$a\leftrightarrow b=/a\rightarrow b/\&/b\rightarrow a/ \qquad (6)$$

$a,b\in[0,1]$, where implication "$\rightarrow$" and conjunction "&" are modelled by φ-operator and t-norm, respectively, i.e.

$$a\leftrightarrow b=(a\varphi b)\ t\ (b\varphi a)\ . \qquad (7)$$

The minimum taken over the whole family of the fuzzy sets $\hat{B}^{(k)}$ and $B^{(k)}$ corresponds to the global view of the accuracy achieved by the constructed model.
Take into account the output space $\mathbb{B}$ as the universe of discourse. One is

interested in evaluation of the quality/relevancy/of the model. This evaluation is specified by expressing a grade of satisfaction of the property

"the space $\mathbb{B}$ is well-mapped/well-represented/ by the fuzzy model",

This means that for any input A, the respective output of the model and the fuzzy set describing the output of the real process/system/ are indistinguishable, or at least close each other. The vector $\underline{\gamma}$ cannot form the overall characterization of the quality of the model due to the following facts,

/i/ respective $\underline{\gamma}$'s of the two models can be incomparable,

/ii/ each coordinate of $\underline{\gamma}$ realizes a partial evaluation of the quality of the model, viz. conveys an information on each coordinate separately.

The evaluation of the global property mentioned above cannot be deduced by simple aggregation of the partial evaluations of the model/corresponding elements of $\underline{\gamma}$/.

A suitable fuzzy measure will serve as an appropriate tool reflecting the overall performance of the model. By definition, the fuzzy measure/and, of course, the λ-fuzzy measure/is defined as a set function expressed on the Borel field $\mathcal{B}$ of $\mathbb{B}$, which satisfies the following axioms,

/i/ $G_\lambda(\emptyset)=0$, $G_\lambda(\mathbb{B})=1$ boundary conditions

/ii/ if $B_1 \subset B_2$ then $G_\lambda(B_1) \le G_\lambda(B_2)$ monotonicity

/iii/ for any monotone sequence $B_1, B_2, \ldots, B_n, \ldots$

$$\lim_{n\to\infty} G_\lambda(B_n) = G_\lambda(\lim_{n\to\infty} B_n)$$

This last property is important only for infinite sequences of sets. The property of monotonicity is especially interesting. It replaces the more restrictive additivity property characterizing each probability measure.

The fuzzy measure $G_\lambda(.)$ provides us by a quantitative characterization of the property abovementioned. If the number of points of the output space $\mathbb{B}$ increases, at which we know the concrete levels of $\underline{\gamma}$, then also our ability increases to formulate our judgement about the model as a whole. If one does not take into account any point of $\mathbb{B}$, we can judge nothing, i.e. $G_\lambda(\emptyset)=0$. If we have knowledge only about a single point of B, say b_i, this is expressed by $G_\lambda(\{b_i\})$. Then we can only partially evaluate the model-our knowledge is not complete. If you consider only a single b_i, a significant error of overestimation of the quality of the model may occur, if γ_i is nearly equal to 1,0. Contrary, if γ_i=0.0 we can arrive to a pessimistic underevaluation.

From computational point of view having $\underline{\gamma}$ the λ-fuzzy measure $G_\lambda(.)$ can be easily established. The value of λ is obtained by a numerical procedure that resolves the following nonlinear equation with respect to λ,

$$\left(\prod_{i=1}^{m}(1+\lambda\gamma_i)-1\right) / \lambda = 0 \qquad (8)$$

where $\lambda \in (-1, \infty)$.

Let us discuss the property of controllability of the system described by means of the fuzzy model/1/. A goal of control is specified by a fuzzy set $\bar{B}$ in the output space $\mathbb{B}$. We are interested in determining the fuzzy set of input, viz. control, that enables us to attain $\bar{B}$, or, at least, to reach it as close as possible. Having the model of the system, we can solve this problem considering the following equation,

$$A \boxdot \hat{R} = \bar{B} \tag{9}$$

and solving it with regard to A for given $\bar{B}$ and $\hat{R}$. As a straightforward consequence coming from the area of fuzzy relation equations, if $\mathbb{A} = \{A : \mathbb{A} \to [0,1] \mid A \boxdot \hat{R} = \bar{B}\}$ is nonempty, then the fuzzy set of control $\hat{A}$ is equal to

$$\hat{A} = \hat{R} \circledast \bar{B} \tag{10}$$

viz.

$$\hat{A}(a_j) = \min_{1 \le i \le m} [\hat{R}(a_j, b_i) \varphi \bar{B}(b_i)] \tag{11}$$

But if $\mathbb{A} = \emptyset$ one should check to which extent $\hat{A}$ leads to $\bar{B}$ indeed. For this purpose compute $\bar{\bar{B}}$ by

$$\bar{\bar{B}} = \hat{A} \boxdot \hat{R} \tag{12}$$

The/pointwise/degree of equality of $\bar{\bar{B}}$ and $\bar{B}$ is calculated as before as the index $\underline{h}$ specified coordinate-wise as

$$\underline{h} = [h_1 \; h_2 \ldots \; h_m] = [\bar{\bar{B}}(b_1) \longleftrightarrow \bar{B}(b_1) \;\; \bar{\bar{B}}(b_2) \longleftrightarrow \bar{B}(b_2) \ldots \; \bar{\bar{B}}(b_m) \longleftrightarrow \bar{B}(b_m)] \tag{13}$$

The higher the values of $\underline{h}$, the more controllable the fuzzy model. We want to underline that it does not mean exactly the same the system is controllable. This in turn depends on the quality of the fuzzy model; it has the same meaning only in the case of a perfect model. Therefore the fuzzy integral /7/

$$E(\underline{h}) = \oint_{\mathbb{B}} \underline{h} \boxdot G_\lambda(\cdot) \tag{14}$$

$$E(\underline{h}) = \sup_{\alpha \in [0,1]} [\alpha \; t \;\; G(F_\alpha)] \quad , \; F_\alpha = \{b_i \in \mathbb{B} \mid \underline{h}(b_i) \ge \alpha\} \tag{15}$$

" measures" the possible control quality at $\bar{B}$ for the system in terms of two notions, viz.

- the control quality for $\bar{B}$ by the model,
- the overall quality of the model.

Few words of interpretation. Consider two cases:

a. Let $\underline{h} = \underline{1}$/viz. all the elements of $\underline{h}$ are equal to 1.0/This means, with the model we reach the goal $\bar{B}$. The fact whether satisfaction of this property/viz. controllability/is the highest one, depends on the quality of the model. The better the model, the higher $E(\underline{h})$. This is formally expressed by the monotonicity property of the fuzzy integral, i.e.: if $G_\lambda(\cdot) > G'_\lambda(\cdot)$ then $\oint_{\mathbb{B}} \underline{h} \boxdot G_\lambda(\cdot) \ge \oint_{\mathbb{B}} \underline{h} \boxdot G'_\lambda(\cdot)$. Note that without such an evaluation we may arrive to an optimistic statement that the system is controlled without difficulty.But,$\underline{h=1}$ indicates only the fact that the model is well controlled at $\bar{B}$.

b. Let $\underline{h}\neq\underline{1}$.Now the model can be controlled at the state $\bar{B}$ only approximately. Again here $E(\underline{h})$allows us to formulate a more objective judgement whether the system is controllable.

References

1.S.Gottwald,Generalization of some results of Elie Sanchez,BUSEFAL,16,1983, 54-60.

2.S.Gottwald,Characterizations of the solvability of fuzzy equations,EIK/to appear/.

3.W.Pedrycz,Fuzzy relational equations with rtriangular norms and their resolution,BUSEFAL,11,1982,24-32.

4.W.Pedrycz,Applications of fuzzy relational equations for methods of reasoning in presence of fuzzy data,Fuzzy Sets and Systems/to appear/.

5.W.Pedrycz,On generalized fuzzy relational equations and their applications, J.Math.Anal.Appl./to appear/.

6.B.Schweizer,A.Sklar,Statistical metric spaces.Pacific.J.Math.,10,1960,313-334.

7.M.Sugeno,Theory of fuzzy integrals and its applications.Ph.D.thesis,Tokyo Institute of Technology,Tokyo,Japan,1974.

8.M.Sugeno,Fuzzy measures and fuzzy integrals:a survey.In:Fuzzy Automata and Decision Processes/M.M.Gupta,G.N.Saridis,B.R.Gaines eds./North Holland,Amsterdam,1977,pp.89-102.