

## ON INDEXED FUZZY MODELS OF PRODUCTION

Part 2: Some indexed fuzzy economic model.

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Here an indexed fuzzy economic model for two neighbouring fuzzy time-moment is defined. The starting point for our considerations is the Neumann - Gale's model. In this model, as we know, an economic state at given time-moment is described by means of all goods located in this time-moment, i.e. by a vector with nonnegative coordinates. Anyway taking into account informations about the "quality" standart of the economic state in the considered fuzzy time-moment, we indispensable do not meet with the crisp economic state but - in natural way - with a fuzzy state in this fuzzy time. Consequently, we shall also consider transitions from one to the another indexed fuzzy state. Each transition is described by a two - couple with some preference depending on the appropriate preferences of the indexed fuzzy economic states involved in considered transition. This two - couple is called a process.

**Definition 6.1.** Let  $v'$  and  $v''$  are two indexed fuzzy time-moments such that  $v' \cap v'' = \emptyset$ . We say that  $v'$  and  $v''$  are neighbouring indexed fuzzy time-moment if there does not exists an indexed fuzzy time-moment  $v$  such that  $\forall t' \in \text{supp } v', \forall t'' \in \text{supp } v''$  and  $\forall t \in \text{supp } v$   $t' < t < t''$ .

Let  $K_{v'}$  and  $K_{v''}$  denote the convex indexed fuzzy cones (compare Matłoka (1984)) in the two neighbouring indexed fuzzy time-moments  $v'$  and  $v''$ . The elements of these cones are called indexed fuzzy (economic) states. Let us assume that  $\forall t \in T \quad F_t = R^n$  and if  $\mu_{K_{v'}}(f)(t) > 0$  then  $f(t) > 0$  (if  $\mu_{K_{v''}}(f)(t) > 0$  then  $f(t) \geq 0$ ).

**Definition 6.2.** An indexed fuzzy economic model is a convex and closed indexed fuzzy cone  $M_{v'v''} \subset K_{v'} \times K_{v''}$  such that

$$(0_f, g) \notin \text{supp } M_{v'v''} \text{ for } g \neq 0_g .$$

Let us observe that the indexed fuzzy economic model  $M_{v'v''}$  can be described by a superlinear indexed fuzzy multi-valued function,  $a_{v'v''} : K_{v'} \rightsquigarrow P(K_{v''})$  say, such that its graph  $W_{a_{v'v''}}$  is an convex

and closed indexed fuzzy cone and

$(0_f, g) \notin \text{supp } W_{a_{v'v''}} \text{ for } g \neq 0_g$ , (compare Albrycht and Matłoka (1984)).

**Definition 6.3.** Indexed fuzzy technological rate-growth of the process  $(f', f'')$  of  $M_{v'v''}$ ,  $r(f', f'')$  say, is defined by the following formula:  $\forall (t', t'') \in T \times T$

$$r(f', f'')(t', t'') = \sup \{ r : r \cdot \mu_{K_{v'}}(f')(t') \leq \mu_{M_{v'v''}}(f', f'')(t', t'') \}$$

Let us note that if  $\mu_{K_{v'}}(f')(t') \neq 0$  then  $r(f', f'')(t', t'') < \infty$ , and if  $\mu_{K_{v'}}(f')(t') = 0$  then  $r(f', f'')(t', t'') = \infty$ .

**Definition 6.4.** The -technological rate-growth of the process  $(f', f'')$  of  $M_{v'v''}$ ,  $r(f', f'')$  say, is defined by the following formula:  $\forall (t', t'') \in T \times T$

$$\begin{aligned} r(f', f'')(t', t'') &= \sup \{ r : r f'(t') \leq \mu_{K_v}(f')(t') \leq \\ &\leq f''(t''), \mu_{M_{v''}}(f', f'')(t', t'') \}. \end{aligned}$$

**Definition 6.5.** Technological rate-growth of the process  $(f', f'')$  of  $M_{v'v''}$ ,  $r(f', f'')$  say, is defined by the following formula:  $\forall (t', t'') \in T \times T$

$$r(f', f'')(t', t'') = \sup \{ r : r f'(t') \leq f''(t'') \},$$

where  $(f', f'') \in \text{supp } M_{v'v''}$ .

Let  $(f', f'') \in \text{supp } M_{v'v''}$ . Then the symbol  $(f', f'') = 0$  denotes that  $\forall (t', t'') \in \text{supp } v' \times \text{supp } v'' \quad (f'(t'), f''(t'')) = (0, 0)$ .

**Definition 6.6.** The function

$$r(M_{v'v''}) = \max_{(f', f'') \in \text{supp } M_{v'v''}} r(f', f'')$$

$$(f', f'') \neq 0$$

is called indexed fuzzy technological, resp. -technological, resp. technological rate growth of the indexed fuzzy economic model  $M_{v'v''}$ , where  $r(f', f'')$  is defined by Definition 6.3, resp. 6.4, resp. 6.5.

In what follows it is assumed that the functions  $\mu_{K_v}$  and  $\mu_{M_{v''}}$  are continuous.

**Theorem 6.1.** There exists a process  $(\bar{f}', \bar{f}'')$  such that  $r(\bar{f}', \bar{f}'') = r(M_{v'v''})$ .

**Proof.** At first let us note that for all  $(t', t'') \in \text{supp } v' \times \text{supp } v''$  the function  $r(t', t'') : (f', f'') \rightarrow r(f', f'')(t', t'')$  is upper semicontinuous.

Now, let us note that for any  $w > 0$

$$r(f', f'') = r(w \cdot f', w \cdot f'')$$

and

$$\begin{aligned}
 & \max_{(f', f'') \in \text{supp } M_{v' v''}} r(f', f'')(t', t'') = \\
 & (f'(t'), f''(t'')) \neq 0 \\
 & = \max_{(f', f'') \in \text{supp } M_{v' v''}} r(f', f'')(t', t'') \\
 & \|f'(t'), f''(t'')\| = 1
 \end{aligned}$$

Therefore for all  $(t', t'') \in \text{supp } v' \times \text{supp } v''$  there exists  $r(M_{v' v''})(t', t'')$  such that

$$\begin{aligned}
 r(M_{v' v''})(t', t'') = & \max_{(f', f'') \in \text{supp } M_{v' v''}} r(f', f'')(t', t'') \\
 & (f'(t'), f''(t'')) \neq 0
 \end{aligned}$$

i.e. there exists a process  $(\bar{f}', \bar{f}'')$  such that  $r(\bar{f}', \bar{f}'') = r(M_{v' v''})$ .

Having considered so far only the quantity side of the fuzzy model  $M_{v' v''}$  we turn now to the price side. So, let  $p \in (\mathbb{R}_+^n)^\#$  denote a price vector.

**Definition 6.7.** The  $\mu$ -economic rate-growth of the process  $(f', f'')$  of  $M_{v' v''}$ ,  $r_p(f', f'')$  say, is defined as follows:

$$\forall (t', t'') \in T \times T$$

$$\begin{aligned}
 r_p(f', f'')(t', t'') = & \sup \{ r_p : r_p \cdot p(f'(t')) \cdot \mu_{M_{v' v''}}(f')(t') \leq \\
 & \leq p(f''(t'')) \mu_{M_{v' v''}}(f', f'')(t', t'') \}.
 \end{aligned}$$

**Definition 6.8.** The economic rate-growth of the process  $(f', f'') \in \text{supp } M_{v' v''}$ ,  $r_p(f', f'')$  say, is defined as follows:

$$\forall (t', t'') \in T \times T$$

$$r_p(f', f'')(t', t'') = \sup \{ r_p : r_p \cdot p(f'(t')) \leq p(f''(t'')) \}.$$

**Definition 6.9.** The function

$$r_p(M_{v'v''}) = \max_{(f', f'') \in \text{supp } M_{v'v''}} r_p(f', f'')$$

$$(f', f'') \neq 0$$

is called  $p$ -economic resp. economic rate growth of  $M_{v'v''}$  with respect to price vector  $p$ , where  $r_p(f', f'')$  is defined by Definition 6.7 resp. 6.8.

Let  $K_{v'v''}$  denotes an indexed fuzzy subset of  $\bigcap_{t \in T} R_{t,+}$ .

Let us consider a conical indexed fuzzy functional (compare Matzoka (1984))  $L : K_{v'v''} \rightarrow R_{v'v''}$  such that

- for any sequence  $\{f^n\}_{n=1}^{\infty}$  ( $f^n \in \text{supp } K_v'$  or  $f^n \in \text{supp } K_v''$ )  
if  $f^n \rightarrow f^0$  as  $n \rightarrow \infty$  then  $\sup_r \mu_L(f^n, r) \rightarrow \sup_r \mu_L(f^0, r)$ ,
- if  $f \in \text{supp } K_v'$  or  $f \in \text{supp } K_v''$  then  $\sup_r \mu_L(f, r) > 0$ .

An indexed fuzzy functional we can interpret as fuzzy prices in fuzzy time-moments. The fuzzy price denotes that each good  $f(t)$  has in time-moment  $t$  a some value which we have some opinion. It opinion is described by the number from unit interval.

**Definition 6.10.** Fuzzy economic rate-growth of the process  $(f', f'')$  of  $M_{v'v''}$ ,  $r_L(f', f'')$  say, is described as follows:

$$\begin{aligned} r_L(f', f'')(t', t'') &= \sup \{ r_L : r_L \cdot (\sup_r \mu_L(f', r)(t')) \wedge \mu_{M_{v'v''}}(f'')(t'') \leq \\ &\leq \sup_r \mu_L(f'', r)(t'') \cdot \mu_{M_{v'v''}}(f', f'')(t', t'') \}. \end{aligned}$$

**Definition 6.11.** The function

$$r_L(M_{v'v''}) = \max_{(f', f'') \in \text{supp } M_{v'v''}} r_L(f', f'')$$

$$(f', f'') \neq 0$$

is called fuzzy economic rate-growth of  $M_{v'v''}$  with respect to fuzzy prices  $L$ .

Theorem 6.2. There exist the processes  $(f', f'')$  and  $(g', g'')$  such that

$$r_L(f', f'') = r_L(M_{V^*V^{**}}) \quad \text{and} \quad r_p(g', g'') = r_p(M_{V^*V^{**}}).$$

The proof is quite similar to the proof of Theorem 6.1.

Definition 6.12. It is said that 3-tuple  $(r, p, (\bar{F}', \bar{F}''))$ , where  $r$  denotes a positive function,  $p$  - price vector,  $(\bar{F}', \bar{F}'')$  - a process of  $M_{V^*V^{**}}$  describes  $\mu$ -equilibrium state of  $M_{V^*V^{**}}$  if  $\forall (t', t'') \in T \times T$

$$(i) r(t', t'') \cdot f'(t') \cdot \mu_{K_V}(F')(t') \leq F''(t'') \cdot \mu_{M_{V^*V^{**}}}(\bar{F}, \bar{F}'')(t', t'')$$

$$(ii) r(t', t'') \cdot p(f'(t')) \cdot \mu_{K_V}(f')(t') >$$

$$\geq p(F''(t'')) \cdot \mu_{M_{V^*V^{**}}}(\bar{F}, \bar{F}'')(t', t'') \quad \text{for all}$$

$$(f', f'') \in \text{supp } M_{V^*V^{**}}.$$

Definition 6.13. It is said that  $(r, p, (\bar{F}', \bar{F}''))$  describes the equilibrium state of  $M_{V^*V^{**}}$  if  $\forall (t', t'') \in T \times T$

$$(i) r(t', t'') \cdot \bar{F}'(t') \leq \bar{F}''(t''),$$

$$(ii) r(t', t'') \cdot p(f'(t')) > p(F''(t'')), \quad \forall (f', f'') \in \text{supp } M_{V^*V^{**}}.$$

Definition 6.14. It is said that  $(r, L, (\bar{F}', \bar{F}''))$ , where  $r$  denotes a positive function,  $L$  - fuzzy prices,  $(\bar{F}', \bar{F}'')$  - a process of  $M_{V^*V^{**}}$  describes the fuzzy equilibrium state of  $M_{V^*V^{**}}$  if  $\forall (t', t'') \in T \times T$

$$(i) r(t', t'') \cdot \mu_{K_V}(F')(t') \leq \mu_{M_{V^*V^{**}}}(\bar{F}, \bar{F}'')(t', t''),$$

$$(ii) r(t', t'') \cdot (\sup_r \mu_L(f', r)(t') \wedge \mu_{K_V}(f')(t')) >$$

$$\geq \sup_r \mu_L(f'', r)(t'') \wedge \mu_{M_{V^*V^{**}}}(\bar{F}, \bar{F}'')(t', t'')$$

$$\text{for all } (f', f'') \in \text{supp } M_{V^*V^{**}}.$$

In the all cases  $(\bar{F}', \bar{F}'')$  is called a optimal process of the equilibrium state, a function  $r$  is called a rate-growth of equilibrium

state or a rate-growth of the indexed fuzzy model  $M_{V'V''}$ .

Theorem 6.5.  $(r, L, (\bar{F}', \bar{F}''))$  or  $(r, p, (\bar{F}', \bar{F}''))$  is the fuzzy equilibrium state or  $\mu$ -equilibrium state or equilibrium state of  $M_{V'V''}$  iff for any  $(t', t'') \in T \times T$

$$r(t', t'') \in \langle r_L(M_{V'V''})(t', t''), r(\bar{F}', \bar{F}'')(t', t'') \rangle$$

or

$$r(t', t'') \in \langle r_p(M_{V'V''})(t', t''), r(\bar{F}', \bar{F}'')(t', t'') \rangle$$

respectively, where  $r_L(M_{V'V''})$ ,  $r_p(M_{V'V''})$ ,  $r(\bar{F}', \bar{F}'')$  denote fuzzy economic,  $\mu$ -economic or economic or  $\mu$ -technological or indexed fuzzy technological or technological rate-growth of  $M_{V'V''}$  resp. of  $(\bar{F}', \bar{F}'')$ .

Proof. The proof of the all cases goes in the same lines. So, we are going to present the proof for the first case.

Let  $\forall (t', t'') \in T \times T \quad r(t', t'') \in \langle r_L(M_{V'V''})(t', t''), r(\bar{F}, \bar{F}'')(t', t'') \rangle$ . From Definition 6.3 it follows immediately that the condition (i) in the Definition 6.14 holds. On the other hand let us observe that

$$\sup \{ r_L : r_L \cdot (\sup_r \mu_L(f', r)(t') \wedge \mu_{K_V}(f')(t')) \leq \sup_r \mu_L(f'', r)(t'') \wedge \mu_{M_{V'V''}}(f', f'')(t') \} = \inf \{ r_L : r_L \cdot (\sup_r \mu_L(f', r)(t') \wedge \mu_{K_V}(f')(t')) \geq \sup_r \mu_L(f'', r)(t'') \wedge \mu_{M_{V'V''}}(f', f'')(t'') \}.$$

So, from the definition of the fuzzy economic rate-growth of  $M_{V'V''}$  we get the inequality

$$r_L(M_{V'V''})(t', t'') \cdot (\sup_r \mu_L(f', r)(t') \wedge \mu_{K_V}(f')(t')) >$$

$$\sup_r \mu_L(f'', r)(t'') \wedge \mu_{M_{V'V''}}(f', f'')(t', t''), \quad \forall (f', f'') \in \text{supp } M_{V'V''},$$

which holds for each  $r(t', t'') \geq r_L(M_{V'V''})(t', t'')$ . Therefore the condition (ii) of the Definition 6.14 is fulfilled too.

Now, let us assume that  $(r, L, (\bar{x}, \bar{z}''))$  is the fuzzy equilibrium state of  $M_{\bar{v}' \bar{v}''}$ . So, the Definitions 6.3 and 6.11 yields  
 $r(\bar{x}', \bar{z}'')(\bar{t}', \bar{t}'') = \sup \{ r : r \wedge \mu_{K_{\bar{v}''}}(x')(t') \leq \mu_{M_{\bar{v}' \bar{v}''}}(\bar{x}', \bar{z}'')(\bar{t}', \bar{t}'') \}$

From this equality it follows

$$r(\bar{t}', \bar{t}'') \leq r(\bar{x}', \bar{z}'')(\bar{t}', \bar{t}'') \quad (\#)$$

On the other hand we have

$$\begin{aligned} r_L(M_{\bar{v}' \bar{v}''})(\bar{t}', \bar{t}'') &= \max_{(x', z'') \in \text{supp } M_{\bar{v}' \bar{v}''}} (\inf \{ r_L : r_L \cdot (\sup_r \\ &\quad (x', z'') \neq 0 \\ &\quad \mu_L(x', r)(t') \wedge \mu_{K_{\bar{v}''}}(x')(t') \geq \sup_r \mu_L(x'', r)(t'') \wedge \\ &\quad \wedge \mu_{M_{\bar{v}' \bar{v}''}}(x', z'')(t', t'') \}) \end{aligned}$$

and hence

$$r(\bar{t}', \bar{t}'') \geq r_L(M_{\bar{v}' \bar{v}''})(\bar{t}', \bar{t}'') \quad (\# \#)$$

The inequalities  $(\#)$  and  $(\# \#)$  prove that

$$r(\bar{t}', \bar{t}'') \in \langle r_L(M_{\bar{v}' \bar{v}''})(\bar{t}', \bar{t}''), r(\bar{x}', \bar{z}'')(\bar{t}', \bar{t}'') \rangle.$$

**Theorem 6.4.** There exists an indexed fuzzy functional  $L = 0$  such that for all  $(x', z'') \in \text{supp } M_{\bar{v}' \bar{v}''}$  and all  $(t', t'') \in T \times T$

$$\begin{aligned} \sup_r \mu_L(x'', r)(t'') \wedge \mu_{M_{\bar{v}' \bar{v}''}}(x', z'')(t', t'') &\leq \\ &\leq r(M_{\bar{v}' \bar{v}''})(t', t'') \cdot (\sup_r \mu_L(x', r)(t') \wedge \mu_{K_{\bar{v}''}}(x')(t')) , \end{aligned}$$

where  $r(M_{\bar{v}' \bar{v}''})$  denotes the indexed fuzzy rate-growth of  $M_{\bar{v}' \bar{v}''}$ .

**Proof.** Let us define an indexed fuzzy functional  $L$  in the following way

$$\mu_L(x, r) = \begin{cases} \mu_{K_{\bar{v}''}}(x) & \text{if } x \in \text{supp } K_{\bar{v}''} , \\ \mu_{M_{\bar{v}' \bar{v}''}}(x) & \text{if } x \in \text{supp } M_{\bar{v}' \bar{v}''} , \\ 0 & \text{otherwise ,} \end{cases} \quad r \in \text{supp } R_{\bar{v}' \bar{v}''}$$

Let us note that for this fuzzy functional the inequality in the Theorem 6.4 holds.

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