ON INTERVALS DEFINED BY FUZZY PREFERENCE RELATION

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This paper contains the considerations of the system of fuzzy arrangement (SFA) defined in [2] . Used here the notions of consistence and monotonicity of fuzzy relation FR are defined in [2] , too.

1. Fuzzy intervals and their based properties

Let be given any monotonical(FR) $\eta : \hat{R}^2 \rightarrow [0,1]$ and $\psi : \hat{R}^2 \rightarrow [0,1]$ where the symbol $\hat{R}$ denotes the set of all real numbers with the minus infinity and the infinity.

Definition 1.1: Each mapping $\varphi\langle a,b\rangle : \hat{R} \rightarrow [0,1]$ defined by the identity

$$\varphi\langle a,b\rangle(x) = \psi(a,x) \wedge \eta(x,b) \quad . \qquad (1.1)$$

for every $(a,b,x) \in \hat{R}^3$, is called a fuzzy interval (FI).

Let us consider any SFA $(\varrho, \varrho_e, \varrho_s)$ where ϱ is FR "less or equal" (FLE) and ϱ_s is FR "less than" (FLT). We note, that the identity (1.1) describes all kinds of intervals on real line $\hat{R}$ generalized for fuzzy case, because we have:

- if $\psi = \varrho$ and $\eta = \varrho$ then $\varphi\langle a,b\rangle = \varphi[a,b]$;
- if $\psi = \varrho$ and $\eta = \varrho_s$ then $\varphi\langle a,b\rangle = \varphi[a,b[$;

- if $\psi = \varsigma_s$ and $\eta = \varsigma$ then $\varphi\langle a,b\rangle = \varphi\,]a,b]$;
- if $\psi = \varsigma_s$ and $\eta = \varsigma_s$ then $\varphi\langle a,b\rangle = \varphi\,]a,b[$.

By monotonicity of FR we can to prove that:

Theorem 1.1: We have

$$a \geqslant c \quad \text{and} \quad b \leqslant d \Rightarrow \varphi\langle a,b\rangle \leqslant \varphi\langle c,d\rangle \ ; \tag{1.2}$$

$$\varphi\langle a,b\rangle \wedge \varphi\langle c,d\rangle = \varphi\langle a\vee c, b\wedge d\rangle \ ; \tag{1.3}$$

$$\varphi\langle a,b\rangle \vee \varphi\langle c,d\rangle \leqslant \varphi\langle a\wedge c, b\vee d\rangle \ ; \tag{1.4}$$

for any $(a,b,c,d) \in \hat{R}^4$.

Furthermore, by monotonicity of SFA, we obtain.

Theorem 1.2: We have

$$\varphi\,]a,b[\ \leqslant \varphi[a,b[\ \leqslant \varphi[a,b] \leqslant \varphi[a,d[\ \leqslant \varphi\,]c,d[\ ; \tag{1.5}$$

$$\varphi\,]a,b[\ \leqslant \varphi\,]a,b] \leqslant \varphi[a,b] \leqslant \varphi\,]c,b] \leqslant \varphi\,]c,d[\tag{1.6}$$

for any $(a,b,c,d) \in \hat{R}^4$ such that $a > c$ and $b < d$.

Let us consider such ordered pairs of FR (η,ψ) that

$$\psi = \dot{\overline{\eta^{-1}}} \tag{1.7}$$

The consistence of SFA implies that the pairs (ς, ς_s) and (ς_s, ς) fulfil the condition (1.7). Therefore we mark off the following class of FI.

Definition 1.2: Each FI obtained by means of the pair (η,ψ) fulfilling (1.7) is called half-open FI (HFI).

For any HFI we obtain the following conclusion.

Theorem 1.3: For any $(a,b,c,d) \in \hat{R}^4$, if $b \leqslant c$ or $a \geqslant d$ then the HFI $\varphi\langle a,b\rangle$ and $\varphi\langle c,d\rangle$ are W-separated.

Proof: If $b \leqslant c$ then

$$\varphi\langle a,b\rangle = \psi(a,\cdot)\wedge\eta(\cdot,b) \leqslant \eta(\cdot,b) = 1 - \psi(b,\cdot) \leqslant$$
$$\leqslant 1 - \psi(c,\cdot) \leqslant 1 - \psi(c,\cdot)\wedge\eta(\cdot,d) = 1 - \varphi\langle c,d\rangle$$

By analogous way as above, we check W-separatity between $\varphi\langle a,b\rangle$ and $\varphi\langle c,d\rangle$ for case $a \geqslant d$. ∎

Theorem 1.4: The HFI $\varphi\langle a,b\rangle$ is W-empty set for any pair $(a,b) \in \hat{R}^2$ such that $a \geqslant b$.

Proof: $$\varphi\langle a,b\rangle = \psi(a,\cdot)\wedge\eta(\cdot,b) \leqslant \eta(\cdot,b) = 1 - \psi(b,\cdot)$$
$$1 - \psi(a,\cdot) \leqslant 1 - \psi(a,\cdot)\wedge\eta(\cdot,b) = 1 - \varphi\langle a,b\rangle \text{ . ∎}$$

Definition 1.3: If finite or infinite sequence of fuzzy subsets $\{\nu_n\}$ fulfils the next properties:

- fuzzy subsets ν_n are pairwise W-separated;
- the fuzzy subset $\mu \wedge (1 - \sup_n \{\nu_n\})$ is W-empty set;
- $\sup_n \{\nu_n\} \leqslant \mu$

for fixed fuzzy subset μ then it is called a repartition of μ. [3]

Theorem 1.5: Let be given the finite nondecreasing sequence of numbers $\{a_k\}_{k=1}^n$. Then the sequence of HFI $\{\varphi\langle a_k, a_{k+1}\rangle\}_{k=1}^{n-1}$ is a repartition of HFI $\varphi\langle a_1, a_n\rangle$.

Proof: The thesis follows immediately from

Lemma 1.1: Let be given the finite nondecreasing sequence of fuzzy subsets $\{\mu_k\}_{k=1}^n$. Then the sequence $\{\mu_{k+1}\wedge(1 - \mu_k)\}_{k=1}^{n-1}$ is a repartition of the fuzzy subset $\mu_n \wedge (1 - \mu_1)$. [3]

Moreover, in special case we have

Theorem 1.6: Let the SFA be strict in sense given in [2]. Then HFI $\varphi[a,b[$ and $\varphi]a,b]$ are W-empty iff $a \geqslant b$.

Proof: If $a \geqslant b$ then by the Theorem 1.4 we get that $\varphi[a,b[$ and $\varphi]a,b]$ are W-empty sets. Let us assume that $a<b$. Then by monotonicity of SFA and by quasi-antisymmetry of FLE (see [2]) we obtain $\varphi[a,b[(x) = \varsigma(a,x) \wedge \varsigma_s(x,b) > 0.5$ and $\varphi]a,b](x) = \varsigma_s(a,x) \wedge \varsigma(x,b) > 0.5$ for every $x \in]a,b[$. So, HFI $\varphi[a,b[$ and $\varphi]a,b]$ are not W-empty sets. This fact puts an end to proof.■

2. The fuzzy finite Borel family

Let FI be generated by SFA with unfuzzily bounding FLE (see [2]). The condition (1.2) proves universality of the following definition.

Definition 2.1: If $\widehat{\mathcal{B}}_\varsigma = \{\mu : \hat{R} \to [0,1]\}$ is a family of all such fuzzy subsets μ that

$$\mu = \mu_1 = \max_{k \leqslant n} \{\varphi[a_k, b_k[\} \qquad (2.1)$$

or

$$\mu = \mu_2 = \max_{k \leqslant n} \{\varphi[a_k, b_k[\} \vee \varphi[a_{n+1}, +\infty] \qquad (2.2)$$

where $\{a_k\}$ and $\{b_k\}$ are increasing sequences of numbers in $\hat{R}$, then $\widehat{\mathcal{B}}_\varsigma$ is called a fuzzy finite Borel family.

Definition 2.2: If fuzzy algebra (σ-algebra) $\sigma = \{\mu : X \to [0,1]\}$ [1] does not contain the fuzzy subset $[\![\frac{1}{2}]\!]_X : X \to \{\frac{1}{2}\}$ then it is called a soft fuzzy algebra (σ-algebra). [3]

Lemma 2.1: Let be given any nondecreasing sequences of fuzzy subsets $\{\mu_n\}$ and $\{\psi_n\}$ such that the numbers of their elements are equal. Then the sequence $\{\nu_n\}$ defined by the identity $\nu_n = \mu_n \wedge (1 - \psi_n)$ satisfies the following property

$$1 - \max_{k\leqslant n}\{\nu_k\} = \psi_1 \vee \max_{k<n}\{(1-\mu_k)\wedge\psi_{k+1}\}\vee(1-\mu_n). \text{ [3] } \tag{2.3}$$

Theorem 2.1: Any fuzzy finite Borel family is a soft fuzzy algebra. Furthermore, any fuzzy subsets μ_1 and μ_2 defined by (2.1) and (2.2) satisfy

$$1 - \mu_1 = \varphi[-\infty, a_1[\vee \max_{k<n}\{\varphi[b_k, a_{k+1}[\}\vee\varphi[b_n, +\infty], \tag{2.4}$$

$$1 - \mu_2 = \varphi[-\infty, a_1[\vee \max_{k\leqslant n}\{\varphi[b_k, a_{k+1}[\}. \tag{2.5}$$

Proof: By (2.3) we get

$$1 - \mu_1 = \varrho_s(\cdot, a_1)\vee \max_{k<n}\{(1-\varrho(_s\cdot, b_k))\wedge\varrho_s(\cdot, a_{k+1})\}\vee$$

$$\vee(1 - \varrho_s(\cdot, b_n)) =$$

$$= \varrho_s(\cdot, a_1)\vee\max_{k<n}\{\varphi[b_k, a_{k+1}[\}\vee(1-\varrho_s(\cdot, b_n)) =$$

$$= ((1-\varrho_s(\cdot, -\infty))\wedge\varrho_s(\cdot, a_1))\vee\max_{k<n}\{\varphi[b_k, a_{k+1}[\}\vee$$

$$\vee((1-\varrho_s(\cdot, b_n))\wedge\varrho(\cdot, +\infty)) =$$

$$= \varphi[-\infty, a_1[\vee\max_{k<n}\{\varphi[b_k, a_{k+1}[\}\vee\varphi[b_n, +\infty],$$

Nextly, by the de Morgan's Law and (2.4) we obtain

$$1 - \mu_2 = (1 - \max_{k\leqslant n}\{\varphi[a_k, b_k[\})\wedge(1 - \varphi[a_{n+1}, +\infty]) =$$

$$= (\varphi[-\infty, a_1[\vee\max_{k<n}\{\varphi[b_k, a_{k+1}[\}\vee\varphi[b_n, +\infty])\wedge\varrho_s(\cdot, a_{n+1}) =$$

$$= (\varrho_s(\cdot, a_1)\wedge\varrho_s(\cdot, a_{n+1}))\vee\max_{k<n}\{\varrho(b_k, \cdot)\wedge\varrho_s(\cdot, a_{k+1})\wedge\varrho_s(\cdot, a_{n+1})\}\vee$$

$$\vee(\varrho(b_n, \cdot)\wedge\varrho_s(\cdot, a_{n+1})) = \varphi[-\infty, a_1[\vee\max_{k\leqslant n}\{\varphi[b_k, a_{k+1}[\}.$$

So, $\widehat{\beta}_{\varsigma}$ is closed under complement. It is self-evident that is closed under union. Since $\Theta_{\widehat{R}} = \varphi[-\infty,-\infty[\, \in \widehat{\beta}_{\varsigma}$, $\widehat{\beta}_{\varsigma}$ is a fuzzy algebra.

We have $\varphi[a,b[\,(+\infty) = 0$ for any $(a,b) \in \widehat{R}^2$. Therefore, each fuzzy subset defined by (2.1) fulfils $\mu_1(+\infty) = 0$. Moreover, we have $\mu_2(+\infty) = 1$ for any fuzzy subset defined by (2.2) because $\varphi[a,+\infty]\,(+\infty) = 1$ for any $a \in \widehat{R}$. So, $[\![\frac{1}{2}]\!]_{\widehat{R}} \notin \widehat{\beta}_{\varsigma}$. The proof is complete. ∎

3. The fuzzy infinite Borel family

Let the SFA $(\varsigma, \varsigma_e, \varsigma_s)$ is generated by continuous from above FIE such that it unfuzzily bounds the real line (see [2]) . The symbol $F(\widehat{R})$ denotes the family of all fuzzy subsets of $\widehat{R}$.
Since $F(\widehat{R})$ is a fuzzy σ-algebra containing $\widehat{\beta}_{\varsigma}$, there exists the smallest fuzzy σ-algebra containing $\widehat{\beta}_{\varsigma}$.

Definition 3.1: The smallest fuzzy σ-algebra β_{ς} containing $\widehat{\beta}_{\varsigma}$ is called a fuzzy infinite Borel family.

Lemma 3.1: The FI $\varphi\langle a,b\rangle$ defined by (1.1) belongs to β_{ς} for any pairs $(\eta,\psi) \in \{\varsigma, \varsigma_s\}^2$, $(a,b) \in \widehat{R}^2$.
Proof: The Definition 3.1 implies that any FI $\varphi[a,b[\, \in \beta_{\varsigma}$.
Let be given the sequence of numbers in $\widehat{R}$ $\{a_n\}\downarrow a$ and $\{b_n\}\downarrow b$

$$\beta_{\varsigma} \ni \inf_n \{\varphi[a,b_n[\} = \varsigma(a,\cdot) \wedge \inf_n \{\varsigma_s(\cdot,b_n)\} = \varsigma(a,\cdot) \wedge \varsigma(\cdot,b) =$$

$$= \varphi[a,b] ;$$

$$\beta_{\varsigma} \ni \sup_n \{\varphi[a_n,b]\} = \sup_n \{\varsigma(a_n,\cdot)\} \wedge \varsigma(\cdot,b) = \varsigma_s(a,\cdot) \wedge \varsigma(\cdot,b) =$$

$$= \varphi\,]a,b] ;$$

β_ς, $\sup_n \{\varphi\,[a_n, b\,[\,\} = \sup_n \{\varsigma(a_n, \cdot)\} \wedge \varsigma_s(\cdot, b) = \varsigma_s(a, \cdot) \wedge \varsigma_s(\cdot, b) =$
$= \varphi\,]\,a, b\,[\,$. ■

Theorem 3.1: Any fuzzy infinite Borel family is a soft fuzzy σ-algebra. Furthermore, each fuzzy subset μ in β_ς can be described by identities

$$\mu = \sup_k \{\varphi\,[a_k, b_k\,[\,\} \tag{3.1}$$

or

$$\mu = \sup_k \{\varphi\,[a_k, b_k\,[\,\} \vee \varphi\,[a_0, +\infty] \tag{3.2}$$

where $\{(a_k, b_k)\}$ is a finite or infinite sequence of pairs in $\hat{R}^2$.

Proof: Let the symbol Φ_ς denotes the family of all fuzzy subsets defined by (3.1) or (3.2). Additionally, let us indicate $\overline{\Phi}_\varsigma =$ $= \{\mu : 1 - \mu \in \Phi_\varsigma\}$.

From the Definition 2.1 we get that each $\mu \in \hat{\beta}_\varsigma$ can be described by (3.1) or (3.2). Also $\hat{\beta}_\varsigma$ is closed under complement. Therefore, we have $\hat{\beta}_\varsigma \subset \Phi_\varsigma \cap \overline{\Phi}_\varsigma$. This fact along with the Definition 3.1 implies that $\beta_\varsigma \subset \Phi_\varsigma \cap \overline{\Phi}_\varsigma$ because $\Phi_\varsigma \cap \overline{\Phi}_\varsigma$ is a fuzzy σ-algebra. On the other side, by the Lemma 3.1 we obtain $\Phi_\varsigma \subset \beta_\varsigma$. If we take into account all above facts, then we see that $\Phi_\varsigma \cap \overline{\Phi}_\varsigma \subset \Phi_\varsigma \subset \beta_\varsigma \subset \Phi_\varsigma \cap \overline{\Phi}_\varsigma$. So, $\beta_\varsigma = \Phi_\varsigma$. The second thesis is proved.

Since the FLE unfuzzily bounds $\hat{R}$, we have $\varphi\,[a, b\,[\,(+\infty) = 0$ and $\varphi\,[a, +\infty]\,(+\infty) = 1$ for any pair $(a, b) \in \hat{R}^2$. So, $[\![\tfrac{1}{2}]\!]_{\hat{R}} \notin \beta_\varsigma$. ■

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