

Some Model Fuzzy Distributions Induced
By Random Intervals and Their Computations

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Abstract

Some model fuzzy distribution induced by random intervals are defined and their computation programs are given by Basic + in this paper.

1. The fundamental formulas

Suppose that η_1 and η_2 are independent random variables. The projectable fuzzy set S induced by η_1 and η_2 was denoted by $S(\eta_1, \eta_2)$ in [1]. And the computational formulas of membership function of $S(\eta_1, \eta_2)$ were given as following:

Formula 1: If η_1 and η_2 are independent continuous random variables having distribution functions $F_1(x)$ and $F_2(x)$ or distribution densities $p_1(y)$ and $p_2(y)$, then

$$\begin{aligned}\mu_{S(\eta_1, \eta_2)}(x) &= F_1(x) + F_2(x) - 2F_1(x) \cdot F_2(x) \\ &= \int_{-\infty}^x p_1(y) dy + \int_{-\infty}^x p_2(y) dy - 2 \int_{-\infty}^x p_1(y) dy \cdot \int_{-\infty}^x p_2(y) dy.\end{aligned}$$

Formula 2: If η_1 and η_2 are independent random variables having distribution laws $P(\eta_1 = y_{1i}) = p_{1i}$ and $P(\eta_2 = y_{2j}) = p_{2j}$, then

$$\mu_{S(\eta_1, \eta_2)}(x) = \sum_{y_{1i} \leq x} p_{1i} + \sum_{y_{2j} \leq x} p_{2j} - (\sum_{y_{1i} \leq x} p_{1i}) \cdot (\sum_{y_{2j} \leq x} p_{2j}) - (\sum_{y_{1i} < x} p_{1i}) \cdot (\sum_{y_{2j} < x} p_{2j}).$$

(see [1])

2. Some model fuzzy distribution and their computations

The membership function of $S(\eta_1, \eta_2)$ has been given by formula 1 and formula 2, but their computations are very difficult.

The computation programs of membership functions of $S(\eta_1, \eta_2)$ induced by some common probabilities distributions are given by Basic + .

1). Uniform fuzzy distribution

Definition 1: Suppose that η_1 and η_2 are independent random variables having distribution densities:

$$p_{\eta_1}(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{other} \end{cases} \quad \text{and} \quad p_{\eta_2}(x) = \begin{cases} \frac{1}{d-c}, & c \leq x \leq d \\ 0, & \text{other} \end{cases}$$

then the $\underline{s}(\eta_1, \eta_2)$ is called uniform fuzzy distribution. From formula 1, we have

Proposition 1. If $a < b < c < d$, then $\underline{s}(\eta_1, \eta_2)$ is a fuzzy number, and ^[2]

$$\mu_{\underline{s}(\eta_1, \eta_2)}(x) = \begin{cases} 0, & x < a \quad \text{or} \quad x > d \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b < x < c \\ \frac{d-x}{d-c}, & c < x < d \end{cases}$$

If $a < c < b < d$, then $\underline{s}(\eta_1, \eta_2)$ is a convex fuzzy set, and ^[2]

$$\mu_{\underline{s}(\eta_1, \eta_2)}(x) = \begin{cases} 0, & x < a \quad \text{or} \quad x > d \\ \frac{x-a}{b-a}, & a < x < c \\ \frac{x-a}{b-a} + \frac{x-c}{d-c} - 2 \frac{x-a}{b-a} \cdot \frac{x-c}{d-c}, & c < x < b \\ \frac{d-x}{d-c}, & b < x < d \end{cases}$$

2). Normal fuzzy distribution

Definition 2: Suppose that η_1 and η_2 are independent random variables and $\eta_1 \sim N(p_1, p_2)$, $\eta_2 \sim N(p_3, p_4)$, then the $\underline{s}(\eta_1, \eta_2)$ is called normal fuzzy distribution. ($N(p_i, p_j)$ expresses normal distribution.)

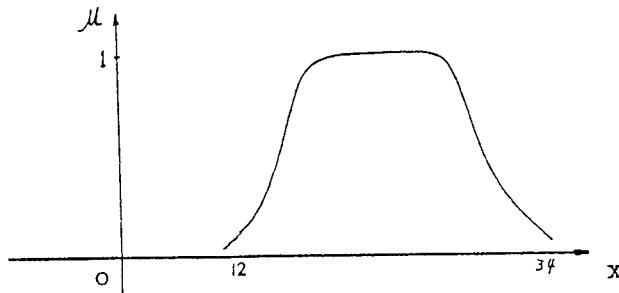
From formula 1, we have

$$\begin{aligned} \mu_{\underline{s}(\eta_1, \eta_2)}(x) = & \int_{-\infty}^x \frac{1}{\sqrt{2\pi} p_2} e^{-\frac{(y-p_1)^2}{2p_2^2}} dy + \int_x^{\infty} \frac{1}{\sqrt{2\pi} p_4} e^{-\frac{(y-p_3)^2}{2p_4^2}} dy \\ & - 2 \int_{-\infty}^x \frac{1}{\sqrt{2\pi} p_2} e^{-\frac{(y-p_1)^2}{2p_2^2}} dy \cdot \int_x^{\infty} \frac{1}{\sqrt{2\pi} p_4} e^{-\frac{(y-p_3)^2}{2p_4^2}} dy \end{aligned}$$

The computation program of $\mu_{\underline{s}(\eta_1, \eta_2)}(x)$ is given by program 1 in the appendix.

Example : Suppose that $\eta_1 \sim N(15, 2^2)$, $\eta_2 \sim N(30, 2^2)$.

If only the input data 15, 4, 30, 4 are provided, the computer can type out the distribution curve of $\mu_{S(\eta_1, \eta_2)}(x)$ immediately as following:



Proposition: If $\eta_1 \sim N(p_1, p_2)$, $\eta_2 \sim N(p_3, p_4)$ and $p_1 < p_3$, then

$$(1) \quad \max \mu_{S(\eta_1, \eta_2)}(x) < 1.$$

$$(2) \quad \text{If } x < p_1, \mu_{S(\eta_1, \eta_2)}(x) \text{ is increasing and } \mu_{S(\eta_1, \eta_2)}(p_1) = \frac{1}{2}$$

$$\text{If } x > p_3, \mu_{S(\eta_1, \eta_2)}(x) \text{ is decreasing and } \mu_{S(\eta_1, \eta_2)}(p_3) = \frac{1}{2},$$

$$(3) \quad \text{If } 0 < \lambda < \frac{1}{2}, S(\eta_1, \eta_2)_\lambda \supset [p_1, p_3]$$

$$\text{If } \frac{1}{2} < \lambda < 1, S(\eta_1, \eta_2)_\lambda \subset [p_1, p_3].$$

3). Exponential fuzzy distribution

Definition 3: Suppose that η_1 and η_2 are independent random variables having distribution densities:

$$p_1(y) = \begin{cases} c \cdot e^{-cy}, & y > 0 \\ 0, & y \leq 0 \end{cases}, \quad p_2(y) = \begin{cases} d \cdot e^{-dy}, & y > 0 \\ 0, & y \leq 0 \end{cases},$$

then the $S(\eta_1, \eta_2)$ is called exponential fuzzy distribution. From formula 1, we have

$$\mu_{S(\eta_1, \eta_2)}(x) = \begin{cases} e^{-cx} + e^{-dx} - 2e^{-(c+d)x}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

The computation program of $\mu_{S(\eta_1, \eta_2)}(x)$ is given by program 2 in the appendix.

4). $\chi^2(m)$ fuzzy distribution

Definition 4: Suppose that η_1 and η_2 are independent random variables and $\eta_1 \sim \chi^2(m_1)$, $\eta_2 \sim \chi^2(m_2)$, then

$S(\eta_1, \eta_2)$ is called $\chi^2(m_1, m_2)$ fuzzy distribution. From formula 1, we have

$$\begin{aligned}\mu_{S(\eta_1, \eta_2)}(x) &= \int_0^x \frac{1}{2^{\frac{m_1}{2}} \Gamma(\frac{m_1}{2})} y^{\frac{m_1}{2}-1} e^{-\frac{1}{2}y} dy + \int_0^x \frac{1}{2^{\frac{m_2}{2}} \Gamma(\frac{m_2}{2})} y^{\frac{m_2}{2}-1} e^{-\frac{1}{2}y} dy \\ &\quad - 2 \int_0^x \frac{1}{2^{\frac{m_1+m_2}{2}} \Gamma(\frac{m_1+m_2}{2})} y^{\frac{m_1+m_2}{2}-1} e^{-\frac{1}{2}y} dy\end{aligned}$$

The computation program of $\mu_{S(\eta_1, \eta_2)}(x)$ is given by program 3 in the appendix.

5). Two points fuzzy distribution

Definition 5: Suppose that η_1 and η_2 are independent random variables having distribution laws

$$\begin{array}{c|cc} \eta_1 & a_1, & b_1 \\ \hline p & 1-p_1, & p_1 \end{array} \quad \text{and} \quad \begin{array}{c|cc} \eta_2 & a_2, & b_2 \\ \hline p & 1-p_2, & p_2 \end{array}, \quad (a_1 < b_1, \quad a_2 < b_2)$$

then $S(\eta_1, \eta_2)$ is called two points fuzzy distribution. From formula 2, we have that

$$(1) \text{ If } a_1 < b_1 \leq a_2 < b_2, \quad \mu_{S(\eta_1, \eta_2)}(k) = \begin{cases} 1-p_1, & k=a_1 \\ 1, & k=b_1 \text{ or } k=a_2 \\ p_2, & k=b_2 \end{cases}$$

$$(2) \text{ If } a_1 < a_2 \leq b_1 < b_2, \quad \mu_{S(\eta_1, \eta_2)}(k) = \begin{cases} 1-p_1, & k=a_1 \\ 1, & k=a_2 \text{ or } k=b_1 \\ p_2, & k=b_2 \end{cases}$$

$$(3) \text{ If } a_1 = a_2 < b_1 = b_2, \quad \mu_{S(\eta_1, \eta_2)}(k) = \begin{cases} 1-p_1 \cdot p_2, & k=a_1 = a_2 \\ p_1 + p_2 - p_1 \cdot p_2, & k=b_1 = b_2 \end{cases}$$

$$(4) \text{ If } a_2 < a_1 < b_1 < b_2, \quad \mu_{S(\eta_1, \eta_2)}(k) = \begin{cases} 1-p_2, & k=a_2 \\ 1-p_1 \cdot p_2, & k=a_1 \\ p_1 + p_2 - p_1 \cdot p_2, & k=b_1 \\ p_2, & k=b_2 \end{cases}$$

Obviously, $S(\eta_1, \eta_2)$ is a convex fuzzy set.

6). Geometry fuzzy distribution

Definition 6: Suppose that η_1 and η_2 are independent random variables having distribution laws:

$$p_{1,i} = (1-p_1) p_1^{i-1}, \quad p_{2,j} = (1-p_2) p_2^{j-1}, \quad i, j = 1, 2, 3, \dots,$$

then $S(\eta_1, \eta_2)$ is called geometry fuzzy distribution. From formula 2,

we have

$$\begin{aligned}\mu_{S(\eta_1, \eta_2)}^{(k)} &= \sum_{i=1}^k (1-p_1)p_1^{i-1} + \sum_{j=1}^k (1-p_2)p_2^{j-1} - \left(\sum_{i=1}^k (1-p_1)p_1^{i-1} \right) \left(\sum_{j=1}^k (1-p_2)p_2^{j-1} \right) \\ &\quad - \left(\sum_{i=1}^{k-1} (1-p_1)p_1^{i-1} \right) \cdot \left(\sum_{j=1}^{k-1} (1-p_2)p_2^{j-1} \right), \quad k=1, 2, 3, \dots\end{aligned}$$

The computation program of $\mu_{S(\eta_1, \eta_2)}^{(k)}$ is given by program 4 in the appendix.

7). Binomial fuzzy distribution

Definition 7: Suppose that η_1 and η_2 are independent random variables having distribution laws:

$$p_{1i} = C_n^i \cdot p_1^i \cdot (1-p_1)^{n-i}, \quad p_{2j} = C_n^j \cdot p_2^j \cdot (1-p_2)^{n-j}, \quad i, j=0, 1, 2, \dots,$$

then $S(\eta_1, \eta_2)$ is called Binomial fuzzy distribution. From formula 2, we have

$$\begin{aligned}\mu_{S(\eta_1, \eta_2)}^{(k)} &= \sum_{i=0}^k C_n^i p_1^i (1-p_1)^{n-i} + \sum_{j=0}^k C_n^j p_2^j (1-p_2)^{n-j} - \sum_{i=0}^k C_n^i p_1^i (1-p_1)^{n-i} \cdot \sum_{j=0}^k C_n^j p_2^j (1-p_2)^{n-j} \\ &\quad - \sum_{i=0}^{k-1} C_n^i p_1^i (1-p_1)^{n-i} \cdot \sum_{j=0}^{k-1} C_n^j p_2^j (1-p_2)^{n-j}, \quad k=0, 1, 2, \dots\end{aligned}$$

The computation program of $\mu_{S(\eta_1, \eta_2)}^{(k)}$ is given by program 5 in the appendix.

8). Poisson fuzzy distribution

Definition 8: Suppose that η_1 and η_2 are independent random variables having distribution laws:

$$p_{1i} = e^{-a} \frac{a^i}{i!}, \quad p_{2j} = e^{-b} \frac{b^j}{j!}, \quad i, j=0, 1, 2, 3, \dots,$$

then $S(\eta_1, \eta_2)$ is called poisson fuzzy distribution. From formula 2, we have

$$\begin{aligned}\mu_{S(\eta_1, \eta_2)}^{(k)} &= \sum_{i=0}^k e^{-a} \frac{a^i}{i!} + \sum_{j=0}^k e^{-b} \frac{b^j}{j!} - \left(\sum_{i=0}^k e^{-a} \frac{a^i}{i!} \right) \left(\sum_{j=0}^k e^{-b} \frac{b^j}{j!} \right) \\ &\quad - \left(\sum_{i=0}^{k-1} e^{-a} \frac{a^i}{i!} \right) \cdot \left(\sum_{j=0}^{k-1} e^{-b} \frac{b^j}{j!} \right), \quad k=0, 1, 2, \dots\end{aligned}$$

The computation program of $\mu_{S(\eta_1, \eta_2)}^{(k)}$ is given by program 6 in the appendix.

APPENDIXProgram 1

```

10  input p1,p2,p3,p4
20  print "p1=";p1,"p2=";p2,"p3=";p3,"p4=";p4
30  print
40  def fna(x)=exp(-(x-p1)*(x-p1)/(2*p2**2))/(sqr(2*pi)*p2)
50  def fnb(x)=exp(-(x-p3)*(x-p3)/(2*p4**2))/(sqr(2*pi)*p4)
60  s1=0
70  s2=0
80  d=1e-02
90  m=2000
100 for j=p1-10 to p3+10
110 a=j
120 b=j+1
130 gosub 300
140 s1=s1+i1
150 gosub 700
160 s2=s2+i2
170 s=s1+s2-2*s1*s2
180 print b;tab(10+40*s);"*";s
190 next j
200 end
300 h1=(b-a)/2.0
310 t1=0
320 n1=1
330 e1=fna(a)+fna(b)
340 f1=fna(a+h1)
350 i3=h1*(e1+4*f1)/3.0
360 h1=h1/2.0
370 n1=2*n1
380 t1=t1+f1
390 f1=0
400 k1=a+h1
410 for j1=1 to n1
420 f1=f1+fna(k1)
430 k1=k1+2*h1
440 next j1
450 i1=h1*(e1+4*f1+2*t1)/3.0
460 if n1>m then 600
470 if abs(i3-i1)<d then 500
480 i3=i1
490 goto 360
500 return
600 print "%/%/%/%/%"
610 stop
700 h2=(b-a)/2.0
710 t2=0
720 n2=1
730 e2=fnb(a)+fnb(b)
740 f2=fnb(a+h2)
750 i4=h2*(e2+4*f2)/3.0
760 h2=h2/2.0
770 n2=2*n2

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780 t2=t2+f2
790 f2=0
800 k2=a+h2
810 for j2=1 to n2
820 f2=f2+fnb(k2)
830 k2=k2+2*h2
840 next j2
850 i2=h2*(e2+4*f2+2*t2)/3.0
860 if n2>m then 910
870 if abs(i4-i2)<d then 900
880 i4=i2
890 goto 760
900 return
910 print "%%%%%%%%%"
920 stop

```

Program 2

```

10 input c,d
15 print "c=";c,"d=";d
16 print
20 def fn(a)=exp(-c*x)+exp(-d*x)-2*exp(-(c+d)*x)
30 for j=0 to 5 step 0.1
40 a=j
50 s=fna(a)
60 print a;tab(10+40*s);"*";s
70 next j
80 end

```

Program 3

```

10 input m1,m2
20 print "m1=";m1,"m2=";m2
30 print
40 if int(m1/2)=m1/2 then 100
50 x=sqr(2*pi)
60 for k=0.5 to m1/2-1
70 x=x*k
80 next k
90 go to 140
100 x=1
110 for k=1 to m1/2-1
120 x=x*k
130 next k
140 r1=1/(2**((m1/2)*x))
150 if int(m2/2)=m2/2 then 210
160 y=sqr(2*pi)
170 for k=0.5 to m2/2-1
180 y=y*k
190 next k
200 go to 250
210 y=1
220 for k=1 to m2/2-1

```

```

230  y=y*k
240  next k
250  r2=1/(2**((m2/2)*y))
310  def fna(t)=t**((m1/2-1)*exp(-t/2)
320  def fnb(t)=t**((m2/2-1)*exp(-t/2)
330  s1=0
340  s2=0
350  d=1e-03
360  m=1000
370  for j=0 to 2*m2 step m2/20
380  a=j
390  b=j+m2/20
400  gosub 500
410  s1=s1+i1
420  gosub 730
430  s2=s2+i2
440  s=s1+s2-2*s1*s2
450  print b;tab(10+40*s);***;s
460  next j
470  end
500  h1=(b-a)/2.0
510  t1=0
520  n1=1
530  e1=fna(a)+fna(b)
540  f1=fna(a+h1)
550  i3=r1*h1*(e1+4*f1)/3.0
560  h1=h1/2.0
570  n1=2*n1
580  t1=t1+f1
590  f1=o
600  k1=a+h1
610  for j1=1 to n1
620  f1=f1+fna(k1)
630  k1=k1+2*h1
640  next j1
650  i1=r1*h1*(e1+4*f1+2*t1)/3.0
660  if n1>m then 710
670  if abs(i3-i1)<d then 700
680  i3=i1
690  goto 560
700  return
710  print "%%%%%%%%"
720  stop
730  h2=(b-a)/2.0
740  t2=0
750  n2=1
760  e2=fnb(a)+fnb(b)
770  f2=fnb(a+h2)
780  i4=r2*h2*(e2+4*f2)/3.0
790  h2=h2/2.0
800  n2=2*n2
810  t2=t2+f2
820  f2=o
830  k2=a+h2
840  for j2=1 to n2
850  f2=f2+fnb(k2)
860  k2=k2+2*h2
870  next j2

```

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```

880 i2=r2*h2*(e2+4*f2+2*t2)/3.0
890 if n2>m then 940
900 if abs(i4-i2)<d then 930
910 i4=i2
920 goto 790
930 return
940 print "%%%%%%%%"
950 stop

```

Program 4

```

10 input p1,p2
15 print "p1=";p1,"p2=";p2
20 print
30 a=0
40 b=0
50 c=1/p1
60 d=1/p2
70 r1=1-p1
80 r2=1-p2
100 for k=1 to 20
120 c=c*p1
130 d=d*p2
140 e=r1*r2*a*b
150 a=a+c
160 b=b+d
180 s=r1*a+r2*b-r1*r2*a*b-e
200 print k;tab(10+40*s);""";s
230 next k
240 end

```

Program 5

```

10 input p1,p2,n
15 print "p1=";p1,"p2=";p2,"n=";n
18 print
20 e1=0
30 e2=0
80 for j=0 to n
90 f1=e1
100 f2=e2
120 gosub 300
140 e1=e1+d*p1**j*(1-p1)**(n-j)
150 e2=e2+d*p2**j*(1-p2)**(n-j)
160 s=e1+e2-e1*e2-f1*f2
200 print j;tab(10+40*s);""";s
220 next j
240 end
300 r=j
310 m=n
320 gosub 500
330 a=q
340 m=r
350 gosub 500

```

```

360 b=q
370 m=n-r
380 gosub 500
390 c=q
400 d=a/(b*c)
410 return
500 rem calculate m!
510 q=1
520 if m=0 then 600
530 for k=1 to m
540 q=q*k
550 next k
600 return

```

Program 6

```

10 input a,b
15 print "a=";a,"b=";b
18 print
20 e1=0
30 e2=0
80 for j=0 to 10
90 f1=e1
100 f2=e2
120 m=j
130 gosub 500
140 e1=e1+a**j*exp(-a)/d
150 e2=e2+b**j*exp(-b)/d
160 s=e1+e2-e1*e2-f1*f2
200 print j;tab(10+40*s);"*";s
220 next j
230 end
500 rem calculate m!
510 d=1
520 if m=0 then 600
530 for k=1 to m
540 d=d*k
550 next k
600 return

```

REFERENCES

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- (2) Wang Pei-zhuang, Theory of fuzzy set and its applications, Shanghai Publishing House of Science and Technology.