DISTINCTION BETWEEN FUZZY SETS

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1. Intuitive background.

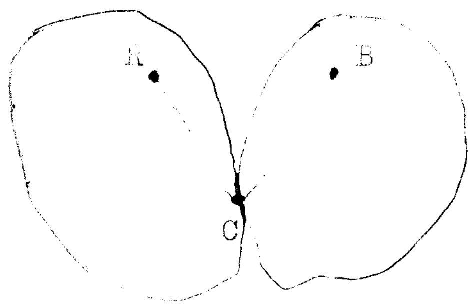
Let us take two fuzzy sets, labeled e.g. "rich" and "beautiful". From the formal point of view the both are functions from some reference set, e.g. the set of people, to the unit interval.
In this case there is no problem in definding the distance or distinction between any two functions.
Fuzzy sets however, are invented not as an abstract theory, but as a formal tool to deal with, or to model real phenomenon.
The measure of distinction should capture therefore some essential properties of reality modeled by fuzzy sets.

How can we interpret reasonably the distance between the property named "rich" and property "beautiful", and not between functions μ_{rich} and $\mu_{\text{beautiful}}$?

By analogy, let us take two beings A and B from different Abbot's flatlands. There is no sense to measure distance between A and B.
We can however compare the distinction between A and B referencing to the ideal, which is the same for all flatlands beings i.e. it is circle.
"Rich" and "beautiful" are also "beings" from different Zadeh's fuzzlands say, Riches and Beauty, belonging possibly to the same fuzzy Realm.
It is rather difficult to find the ideal in this realm, but we know the condition for belonging to it: to be fuzzy.
Some of the "citizens" are even perfectly fuzzy, fuzziness can form

therefore the base for the units of measuring.

In order to define the measure of distinction between two fuzzy sets, we assume that for any two non-comparabile /belonging to the different lands/ fuzzy sets, say A and B, exists some common set C which connects them.



For example, for sets "rich" and "beautiful" the common set could be labeled "rich and beautiful". This set plays the same role of the ideal, both for Riches and for Beauty, like circle for flatlands.

We assume furthermore the existence of some adjacency relation such that for any two fuzzy sets there is the chain of adjacent sets. The distance for two adjacent sets is prespecified according to the goals of investigation and the nature of modeled phenomena. This distance determines the scale of measurement.

2. Basic notions and assumptions.

Let $X = \{x_1, \ldots, x_n\}$ be any set. Fuzzy subset

$$A = \{(x_i, \mu_A(x_i)) \mid x_i \in X,\ \mu_A : X \to [0,1]\}$$

will be identified with n-tuple $a = (a_1, \ldots, a_n)$ where $a_i = \mu_A(x_i)$.

The family of all fuzzy subsets denoted by $F(X)$ is supposed to be partialy ordered. It is assumed that the order is induced by some idempotent, associative and commutative operation $\cap$.

We say that fuzzy set $c \in F(X)$ connects or links fuzzy subsets a and b, and we denote this by $a - c - b$, if either

$$a \cap c = c \quad \text{and} \quad c \cap b = b \quad \text{or}$$

$$a \cap c = c \quad \text{and} \quad c \cap b = c \quad \text{and} \quad a \cap b = c.$$

In the first case we have

$$b \leqslant c \leqslant a,$$

and in the second case

$$c \leqslant a \quad \text{and} \quad c \leqslant b.$$

Subsets $L_i \subset F(X)$, $i=1,\dots,n$ defined by

$$a, b \in L_i \quad \text{iff} \quad a_i \neq b_i \text{ and } a_j = b_j \text{ for } i \neq j$$

are called lines in the space of fuzzy subsets $F(X)$.

We say that sets a and b are adjacent, what is denoted by $a|b$, if they belong to the same line and $a \cap b = a$ or $a \cap b = b$.

3. Aczel operator.

It seems reasonable to require that connection of two indices of real numbers should be treated as an operation which is

- -associative,
- -commutative,
- -monotone and
- -continuous.

Due to Aczel we know that such a operation, which we denote by $\square$, is defined as follows:

$$a \,\square\, b = f\left(f^{-1}(a) + f^{-1}(b)\right)$$

where f is some continuous and strictly monotone real function. Operation $\square$ we call here Aczel operator, and for $a_1 \square a_2 \square \dots \square a_n$ we use the abreviation $\underset{i=1}{\overset{n}{\boxminus}} a_i$,

symbol $\boxminus$ is stilted letter A, the first letter of the words: Aczel, aggregation, addition.

Examples.

Suppose that $f(x) = x$, then $a \,\square\, b = f\left(f^{-1}(a) + f^{-1}(b)\right) = a + b$, and this kind of aggregation could be used for evaluating the truth of sentence "society is rich".

If $f(x) = e^{-x}$, then $f^{-1}(y) = -\ln(x)$, and $a \,\square\, b = a \cdot b$, this kind of aggregation could be used for evaluating the truth of the sentence "Bob and Clara are happy".

4. Distinction measure.

Distinction or dissimilarity measure between two fuzzy sets is [illegible] as some index which:

- is non-negative and for identical sets takes minimal value,
- if c links a and b , then distinction between a and b is defined as a connection of indices of dissimilarity between a and c, and between c and b,
- distinction between the adjacent sets is defined separately /maybe axiomaticaly/.

Formaly, index of dissimilarity is any function

$$D : F(X)\times F(X) \to R$$

satisfying the following conditions:

1. $D(a,b) \geqslant 0, \quad D(a,a) = 0,$
2. $D(a,b) = D(b,a),$
3. $D(a,b) = D(a,c) \sqcap D(c,b)$, if $a - c - b$,
4. $D(a,b) = d_i(a,b)$, if $a,b \in L_i$.

From these conditions follow

$$D(a,b) = \bigsqcap_{i=1}^{n} d_i \; a,b \tag{1}$$

or

$$D(a,b) = f\left(\sum_{i=1}^{n} f^{-1}(d_i(a,b))\right)$$

where f is continuous and strictly monotone function and d_i satisfy the conditions:

$$d_i(a,b) \geqslant 0,\; d_i(a,a) = 0,\; d_i(a,b) = d_i(b,a).$$

In order to see that, let us constuct the following chain:

$$a|c^1|c^2|\ldots|c^{n-1}|b$$

such that

$$c_1^1 = b_1, \quad c_j^1 = a_j, \quad j \neq 1$$

$$c_2^2 = b_2, \quad c_j^2 = c_j^1, \quad j \neq 2$$

$$\vdots$$

$$c_{n-1}^{n-1} = b_{n-1}, \quad c_j^{n-1} = c_j^{n-2}, \quad j \neq n-1,$$

where
$$a = (a_1, \ldots, a_n)$$
$$b = (b_1, \ldots, b_n)$$
$$c^i = (c_1^i, \ldots, c_n^i), \quad i = 1, \ldots, n-1 .$$

It is easy to see that

$$a,\ c^1 \in L_1, \quad c^1,\ c^2 \in L_2, \quad \ldots \quad c^{n-1}, b \in L_n .$$

Since from $a|b|c$ follows $a - b - c$, then we have

$$a - c^1 - c^2 - \ldots - c^{n-1} - b.$$

From condition 4. we have

$$D(a,c^1) = d_1(a,c^1) \quad \ldots \quad D(c^{n-1},b) = d_n(c^{n-1},b)$$

and from condition 3. we have (1) .

Examples

Let $d_i(a,b) = |a_i - b_i|$ and $f(x) = x^r$, then

$$D(a,b) = \sqrt[r]{\sum_{i=1}^{n} |a_i - b_i|^r}$$

In particular

$$D(a,b) = \sum |a_i - b_i| \quad \text{for } r = 1$$

$$D(a,b) = \sqrt{\sum (a_i - b_i)^2} \quad \text{for } r = 2$$

$$D(a,b) = \max |a_i - b_i| \quad \text{for } r \to \infty$$

Let Δ denotes the operator of symmetcal difference, and S is entropy H-function, then taking $d_i(a,b) = S(a_i \Delta b_i)$ we obtain

$$D(a,b) = \varphi(a \Delta b)$$

where φ is measure of fuzziness defined by

$$\varphi(x) = \sum S(x_i)$$

Taking $d_i(a,b) = \varphi(a_i) + \varphi(b_i) - 2\cdot\varphi(c_i)$, where $c_i = a_i \cap b_i$ we have

$$D(a,b) = \varphi(a) + \varphi(b) - 2\cdot\varphi(a \cap b),$$

in particular case, when fuzzy sets are ordered lineary, we have

$$D(a,b) = |\varphi(a) - \varphi(b)|.$$

References

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