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Summary. After some auxiliary lemmas on inequalities in a poset we give a new attempt to classify whether a fuzzified notion is a good extension of a given crisp notion. As <sup>a</sup> example we consider in detail the fuzzification grades for extended union, intersection and Cartesian product of fuzzy sets.

1. Functional inequalities in a poset. We shall consider certain inequalities in a poset  $(L, \leq)$  under different assumptions on  $L$  (cf. Birkhoff [1]). At first let us observe that

Lemma 1. For any  $a, b \in L$  we have

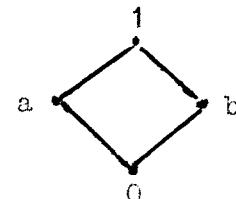
- (1)  $(a \geq t \Rightarrow b \geq t \text{ for } t \in L) \Leftrightarrow a \leq b \Leftrightarrow (b \leq t \Rightarrow a \leq t \text{ for } t \in L)$ ,  
 (1')  $(a \geq t \Leftrightarrow b \geq t \text{ for } t \in L) \Leftrightarrow a = b \Leftrightarrow (b \leq t \Leftrightarrow a \leq t \text{ for } t \in L)$ .

Moreover, if  $L$  is a chain, then

- (2)  $(a > t \Rightarrow b > t \text{ for } t \in L) \Leftrightarrow a \leq b \Leftrightarrow (b < t \Rightarrow a < t \text{ for } t \in L)$ ,  
 (2')  $(a > t \Leftrightarrow b > t \text{ for } t \in L) \Leftrightarrow a = b \Leftrightarrow (b < t \Leftrightarrow a < t \text{ for } t \in L)$ .

Example 1. Let  $L = \{0, a, b, 1\}$  be a poset with the following Hasse diagram (at right):

It is not a chain and the equivalences (2), (2') are not valid in  $L$ . So the additional assumption in Lemma 1 is necessary.



For fixed  $n \geq 2$  let us consider an  $n$ -ary operation  $\phi : L^n \rightarrow L$  with the following properties (for every  $r \in L$  and  $a \in L^n$ ):

- (3)  $\phi(a) \geq r \Leftrightarrow \bigvee_i (a_i \geq r)$ ,  
 (4)  $\phi(a) \leq r \Leftrightarrow \bigvee_i (a_i \leq r)$ ,  
 (5)  $\phi(a) \geq r \Leftrightarrow \bigwedge_i (a_i \geq r)$ ,  
 (6)  $\phi(a) \leq r \Leftrightarrow \bigwedge_i (a_i \leq r)$ ,  
 (7)  $\phi(a) > r \Leftrightarrow \bigvee_i (a_i > r)$ ,  
 (8)  $\phi(a) < r \Leftrightarrow \bigvee_i (a_i < r)$ ,  
 (9)  $\phi(a) > r \Leftrightarrow \bigwedge_i (a_i > r)$ ,  
 (10)  $\phi(a) < r \Leftrightarrow \bigwedge_i (a_i < r)$ ,

where  $a = (a_1, \dots, a_n)$  and  $1 \leq i \leq n$ . We shall prove that the operation  $\phi$  is uniquely determined by any of the above conditions.

<sup>\*)</sup> It is a part of [1], Chap. 2.

Lemma 2.  $1^o$   $\phi$  fulfills (3) iff  $L$  is a meet-semilattice and

$$(11) \quad \phi(a) = \bigwedge_i a_i .$$

$2^o$   $\phi$  fulfills (4) iff  $L$  is a join-semilattice and

$$(12) \quad \phi(a) = \bigvee_i a_i .$$

$3^o$   $\phi$  fulfills (5) iff  $L$  is a chain and (12) holds.

$4^o$   $\phi$  fulfills (6) iff  $L$  is a chain and (11) holds.

$5^o$  If  $L$  is a chain then  $(7) \Leftrightarrow (10) \Leftrightarrow (11)$  and  $(8) \Leftrightarrow (9) \Leftrightarrow (12)$ .

Proof. Conditions (3) - (12) are assumed for every  $a \in L^n$  so we can make different specifications. Let  $s, t \in L$  and  $\phi$  fulfills (3). Putting

$$(13) \quad s * t = \phi(a) \text{ for } a_1 = s, a_2 = \dots = a_n = t$$

we see that

$$s \geq s * t, t \geq s * t, (s \geq r, t \geq r) \Rightarrow s * t \geq r,$$

i.e. there exists  $s \wedge t = s * t$ . Thus  $L$  is a meet-semilattice. But in the meet-semilattice we have (for every  $r \in L$  and  $a \in L^n$ )

$$(3a) \quad \bigwedge_i a_i \geq r \Leftrightarrow \bigvee_i (a_i \geq r)$$

and by Lemma 1  $(11) \Leftrightarrow (3)$ , which proves  $1^o$  and the proof of  $2^o$  is similar.

Now let  $\phi$  fulfil (5). Putting (13) for  $s, t \in L$  we get

$$s * t \geq s, s * t \geq t \text{ and } (s * t \leq s \text{ or } s * t \leq t).$$

So  $t \leq s$  or  $s \leq t$ , i.e.  $L$  is linearly ordered. Thus  $L$  is a chain lattice and (for every  $r \in L$ ,  $a \in L^n$ )

$$(5a) \quad \bigvee_i a_i \geq r \Leftrightarrow \exists_i (a_i \geq r),$$

which implies  $(5) \Leftrightarrow (12)$  by Lemma 1. This proves  $3^o$  and similarly we get  $4^o$ . Using  $1^o - 4^o$  in a chain we see that  $5^o$  is a direct consequence of Lemma 1, which finishes the proof.

Observe that Lemma 2 for  $n=2$  gives different characterizations of meet and join in a poset  $L$ .

Example 2. Using the lattice  $L$  from Example 1 we shall show that the additional assumption in  $5^o$  is necessary. For  $n=2$  we define  $\phi : L^2 \rightarrow L$  by Table 1 or Table 2 where  $p, q, r, s, t, u, v, w \in \{a, b\}$ :

$\emptyset$	0	a	b	1
0	0	0	0	0
a	0	p	q	r
b	0	s	t	u
1	c	v	w	1

Table 1

$\emptyset$	0	a	b	1
0	0	p	q	1
a	r	s	t	1
b	u	v	w	1
1	1	1	1	1

Table 2

Using Table 1 we see that  $\phi$  fulfills (7) and (10) but  $\phi \neq \wedge$ . Similarly,  $\phi$  from Table 2 fulfills (8) and (9) but  $\phi \neq \vee$  (any table defines 256 different binary operations in  $L$ ).

For the generalization of Lemma 2 we introduce another operation  $\phi$ ,  $\phi : 2^L \rightarrow L$  with properties (3') - (12'), where  $a \in L^n$  is changed for  $t \in L$  and  $i$ ,  $a_i$  are changed for  $t \in T$ . E.g. (3) will be changed for  
 $(3') \quad \phi(T) \geq r \Leftrightarrow \bigvee_{t \in T} (t \geq r).$

For a finite  $L$  these properties are reduced to (3) - (12). Therefore we consider the case of an infinite  $L$ .

Lemma 2'. Let  $L$  be an infinite poset.

- $1^\circ$   $\phi$  fulfills (3') iff  $L$  is meet-complete and (11') holds.
- $2^\circ$   $\phi$  fulfills (4') iff  $L$  is join-complete and (12') holds.
- $3^\circ$   $\phi$  fulfills (5') iff  $L$  is a chain with property  
 (\*) any element bounded from below has an immediate predecessor and (12') holds.
- $4^\circ$   $\phi$  fulfills (6') iff  $L$  is a chain with property  
 (\*\*) any element bounded from above has an immediate successor and (11') holds.

Moreover, if  $L$  is a chain, then

- $5^\circ$  We have (9')  $\Leftrightarrow$  (12') and (10')  $\Leftrightarrow$  (11').
- $6^\circ$   $\phi$  fulfills (8') iff  $L$  has property (\*) and (12') holds.
- $7^\circ$   $\phi$  fulfills (7') iff  $L$  has property (\*\*) and (11') holds.

Proof. Let  $T \subseteq L$ . If  $\phi$  fulfills (3') then

$$\bigvee_{t \in T} (\phi(t) \leq r) \text{ and } \bigvee_{t \in T} (r \leq t) \Rightarrow r \leq \phi(T),$$

i.e. there exists  $\bigwedge T = \phi(T)$ . So  $L$  is meet-complete and

$$(5'a) \quad \bigwedge_{t \in T} t \geq r \Leftrightarrow \bigvee_{t \in T} (t \geq r).$$

Thus (3')  $\Leftrightarrow$  (11') by Lemma 1, which gives  $1^\circ$  and similarly we get  $2^\circ$ .

Now let  $\phi$  fulfill (5') and  $s \in L$ . After Lemma 2  $L$  is a chain. So if  
 $(14) \quad T = \{t \in L \mid t < s\}$

and  $T \neq \emptyset$  then  $\phi(T) \in T$ , because the supposition  $\phi(T) \geq s$  leads to a contradiction. Moreover  $\phi(T) = \bigvee T$ , i.e. it is the immediate predecessor of the element  $s$ . In any chain with property (\*) we also have (cf.  $2^\circ$ )

$$(5'a) \quad \bigvee_{t \in T} t \geq r \Leftrightarrow \exists_{t \in T} (t \geq r)$$

for  $T \neq \emptyset$  and we get (5')  $\Leftrightarrow$  (12') by Lemma 1. This proves  $3^\circ$  and the proof of  $4^\circ$  is similar (using  $1^\circ$ ). As a consequence of  $1^\circ$  and  $2^\circ$  we also get  $5^\circ$  (by Lemma 1) in a chain. At last  $6^\circ$  and  $7^\circ$  are implied by  $3^\circ$  and  $4^\circ$  in a chain (also by Lemma 1).

As a consequence of Lemma 2' we get a characterization of operations on countable sequences. Let  $\phi : L^{\mathbb{N}} \rightarrow L$  and (3'') - (12'') denote (3) - (12) with  $a \in L^{\mathbb{N}}$  and  $i \in \mathbb{N}$ .

Lemma 2". 1°  $\phi$  fulfills (3") iff L is meet- $\mathbf{G}$ -complete and (11") holds.

2°  $\phi$  fulfills (4") iff L is join- $\mathbf{G}$ -complete and (12") holds.

3°  $\phi$  fulfills (5") iff L is a chain with property (\*) and (12") holds.

4°  $\phi$  fulfills (6") iff L is a chain with property (\*\*) and (11") holds.

Moreover, if L is a chain, then

5° We have (9")  $\Leftrightarrow$  (12") and (10")  $\Leftrightarrow$  (11").

6°  $\phi$  fulfills (8") iff L has property (\*) and (12") holds.

7°  $\phi$  fulfills (7") iff L has property (\*\*) and (11") holds.

Example 3. Let L denote the lattice from Example 1. If  $\phi : L^E \rightarrow L$  fulfills the conditions:

$$\phi(a) = 0 \Leftrightarrow \bigvee_i (a_i = 0), \quad \phi(a) = 1 \Leftrightarrow \bigvee_i (a_i = 1),$$

$$\phi(a) \neq 0, 1 \Leftrightarrow \bigvee_i (a_i > 0) \text{ and } \bigvee_i (a_i < 1),$$

then it fulfills (7") and (10") but it does not fulfil (11"). Similarly, if  $\phi$  fulfills the conditions

$$\phi(a) = 0 \Leftrightarrow \bigvee_i (a_i = 0), \quad \phi(a) = 1 \Leftrightarrow \bigvee_i (a_i = 1),$$

$$\phi(a) \neq 0, 1 \Leftrightarrow \bigvee_i (a_i > 0) \text{ and } \bigvee_i (a_i < 1),$$

then it fulfills (8") and (9") but it does not fulfil (12"). This shows that the additional assumption in 5° - 7° of Lemma 2' and Lemma 2" is necessary.

Now we assume that  $* : L^2 \rightarrow L$  is an isotone binary operation in L, i.e.

$$(15) \quad a \leq b \Rightarrow (a * c \leq b * c, \quad c * a \leq c * b).$$

As a consequence of (15) we get (cf. Czogała, Drewniak [2])

Lemma 3. If L is a lattice then

$$(16) \quad (t * t \leq t \text{ for } t \in L) \Leftrightarrow (a * b \leq a \vee b \text{ for } a, b \in L),$$

$$(17) \quad (t * t \geq t \text{ for } t \in L) \Leftrightarrow (a * b \geq a \wedge b \text{ for } a, b \in L),$$

$$(18) \quad (t * t = t \text{ for } t \in L) \Leftrightarrow (a \wedge b \leq a * b \leq a \vee b \text{ for } a, b \in L),$$

$$(19) \quad (t * t < t \text{ for } t \in L) \Leftrightarrow (a * b < a \vee b \text{ for } a, b \in L),$$

$$(20) \quad (t * t > t \text{ for } t \in L) \Leftrightarrow (a * b > a \wedge b \text{ for } a, b \in L).$$

Moreover, if L is bounded (with bounds 0 and 1), then

$$(21) \quad (t * 1 \leq t, 1 * t \leq t \text{ for } t \in L) \Leftrightarrow (a * b \leq a \wedge b \text{ for } a, b \in L),$$

$$(22) \quad (t * 0 \geq t, 0 * t \geq t \text{ for } t \in L) \Leftrightarrow (a * b \geq a \vee b \text{ for } a, b \in L).$$

Corollary. In a bounded lattice L we have

$$(23) \quad (t * t \leq t, t * 0 \geq t, 0 * t \geq t \text{ for } t \in L) \Leftrightarrow * = \vee,$$

$$(24) \quad (t * t \geq t, t * 1 \leq t, 1 * t \leq t \text{ for } t \in L) \Leftrightarrow * = \wedge.$$

The above lemmas will be used in consideration of L-fuzzy sets.

2. Grades of fuzzification. Further let  $L$  be a bounded lattice and let  $L(X)$  denote the family of all  $L$ -fuzzy sets in an universe  $X \neq \emptyset$ .

For any crisp set  $K \subseteq X$  we consider a family of level fuzzy sets \*)  
 $r_K \in L(X)$  for  $r \in L$ , where

$$(25) \quad r_K(x) = \begin{cases} r & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x \in X$$

(for  $r = 1$  we get  $\chi_K$  - the characteristic function of  $K$  in  $X$ ).

For any  $L$ -fuzzy set  $A$  we consider the following crisp sets:

- a) cuts  $N_t(A) = \{x \in X \mid A(x) \geq t\}$  for  $t \in L$ ,
- b) strong cuts  $M_t(A) = \{x \in X \mid A(x) > t\}$  for  $t \in L$ ,
- c) endographs  $G(A) = \{(x, t) \in X \times L \mid A(x) \geq t\}$ .

Different mathematical notions are extended on fuzzy sets (cf. Negoita, Ralescu [7]). We will introduce a **metamathematical classification** of fuzzified notions.

Definition. Let  $CN$  denote a crisp notion and  $FN$  - its fuzzification. A fuzzification or a fuzzified notion is:

- a) singular if  $FN$  for characteristic functions of crisp sets does not coincide with  $CN$  for these sets,
- b) ordinary \*\*) if  $FN$  for characteristic functions of crisp sets coincides with  $CN$  for these sets,
- c) uniform if  $FN$  for level fuzzy sets coincides with  $CN$  for suitable crisp sets,
- d) regular if  $FN$  for  $L$ -sets coincides with  $CN$  for their cuts,
- e) s-regular if  $FN$  for  $L$ -sets coincides with  $CN$  for their strong cuts,
- f) g-regular if  $FN$  for  $L$ -sets coincides with  $CN$  for their endographs.

The interpretation of such general definition is not unique. So we specify additionally that "coincidence" means logical equivalence and that for different sets we consider d), e) and c) with constant levels (other interpretation may admit "implies" for "coincides" and different levels in consideration of c) - e)).

By Definition ordinarity is a special case of uniformity ( $r = 1$ ) and in general uniformity is implied by regularity and s-regularity, because, for nontrivial cuts we have

$$\begin{aligned} N_t(r_K) &= K \quad \text{for } 0 < t \leq r, \\ M_t(r_K) &= K \quad \text{for } 0 \leq t < r, \quad r \in L. \end{aligned}$$

However (as we see later), different kinds of regularity are not comparable. It seems that the singularity does not allow for using the same name for  $FN$  as for  $CN$ , because their connections are very weak.

\*) We prefer this name for (25) rather than that of Radecki [8].

\*\*) This notion was first introduced by Drewniak [3].

It was shown in [4] (cf. (7)-(9) in [4]) that the partial order in  $L(X)$  is regular and  $\varepsilon$ -regular as a inclusion of fuzzy sets. After Lemma 1 we see that for a chain  $L$  we get also an  $s$ -regular inclusion. By a simple verification the partial order in  $L(X)$  is an ordinary and uniform inclusion. However, the weak inclusion defined in [6], p.22 as

$$A \prec B \Leftrightarrow (A(x) \leq 0.5 \text{ or } B(x) > 0.5 \text{ for } x \in X)$$

is only an ordinary one (case  $L = [0,1]$ ).

We shall examine the fuzzification quality of other fuzzified notions considered in [4] (generalizing remarks on (15), (16) and (20) - (23) in [4]). First let us consider the extended binary operation

(26)  $(A * B)(x) = A(x) * B(x) \text{ for } A, B \in L(X), x \in X,$   
where  $* : L^2 \rightarrow L$  is given. We ask whether (26) defines union or intersection of fuzzy sets  $A$  and  $B$ .

Theorem 1. Let  $C = A * B$  from (26).

1°  $C$  is an ordinary union iff

$$(27) \quad 0 * 0 = 0, \quad 0 * 1 = 1 * 0 = 1 * 1 = 1,$$

and  $C$  is a singular union iff (27) does not hold.

2°  $C$  is an ordinary intersection iff

$$(28) \quad 0 * 0 = 0 * 1 = 1 * 0 = 0, \quad 1 * 1 = 1,$$

and  $C$  is a singular intersection iff (28) does not hold.

3°  $C$  is a uniform union iff the operation  $*$  is idempotent and has identity equal to 0. The unique isotone uniform union is defined by  $* = \vee$ .

4°  $C$  is a uniform intersection iff the operation  $*$  is idempotent and has zero equal to 0. If  $L = [a,b] \subset \mathbb{R}$ , then the unique continuous associative isotone uniform intersection is defined by  $* = \wedge$ .

5°  $C$  is a regular ( $\varepsilon$ -regular) union iff  $L$  is a chain and  $* = \vee$ .

6°  $C$  is a regular ( $\varepsilon$ -regular) intersection iff  $* = \wedge$ .

7° If  $L$  is a chain, then the unique  $s$ -regular union is defined by  $* = \vee$ , and the unique  $s$ -regular intersection is defined by  $* = \wedge$ .

Proof. First we ask for ordinary unions and intersections. Let

$$(29) \quad 1_K(x) = 1_{\bar{K}}(x) * 1_M(x) \text{ for } K, M \subset X, x \in X.$$

If  $K = L \cup M$ , then (for suitable  $x$ ,  $K$  and  $M$ ) (29) implies (27). On the contrary, under assumption (27), (29) implies that  $K = K \cup M$ . Similarly  $L = M \cap K$  in (29) iff (28) holds, which proves 1° and 2°, because the singularity and ordinarity contradict each other.

Now let

$$(30) \quad r_K(x) = r_{\bar{K}}(x) * r_M(x) \text{ for } K, M \subset X, r \in L, x \in X.$$

$C$  is a uniform union iff (30) is equivalent to  $K = K \cup M$ , and this leads to

$$(31) \quad 0 * 0 = 0, \quad 0 * r = r * 0 = r * r = r \text{ for } r \in L.$$

Since (31) means that the operation \* is idempotent and has identity 0, when we have the first part of 3° and the second part is implied by the Corollary. In the similar proof of 4° the second part is implied by Corollary 4 from paper [2].

By Definition  $C = A * B$  is a regular union iff

$$N_t(C) = N_t(A) \cup N_t(B) \text{ for } t \in L,$$

which leads to the condition

(32)  $A(x) * B(x) \geq t \Leftrightarrow A(x) \geq t \text{ or } B(x) \geq t \text{ for } t \in L, x \in X.$   
Now using Lemma 2, 5° we see that the operation \* fulfills (32) iff L is a chain and  $* = \vee$ . This proves 5°, and 6° is a similar consequence of Lemma 2, 1° (the verification for  $\wedge$ -regularity is the same). The last assertion 7° is a direct consequence of Lemma 2, 5° for  $n = 2$  (cf. (7) and (9)).

Theorem 2. Let

$$(33) \quad \mathcal{C}(A)(x) = C(A(x)) \text{ for } A \in L(X), x \in X,$$

where  $\mathcal{C} : L \rightarrow L$  is given.

1°  $\mathcal{C}(A)$  is an ordinary complement of A iff

$$(34) \quad \mathcal{C}(C) = 1, \quad \mathcal{C}(1) = 0,$$

and it is a singular complement iff (34) does not hold.

2° There exists neither uniform nor regular complement of fuzzy set.

Proof. (33) defines an ordinary complement of A iff

$$1_K = \mathcal{C}(1_X) \Leftrightarrow K = X' \text{ for } K, X \subset X,$$

which is equivalent to (34) and we get 1°. There does not exist a function  $\mathcal{C} : L \rightarrow L$  such that

$$(35) \quad r_{\mathcal{C}} = \mathcal{C}(r_X) \text{ for } r \subset X, r \in L$$

$$(36) \quad N_t^!(A) = N_{\mathcal{C}(t)}(C(A)) \text{ for } A \in L(X), t \in L,$$

because in (35)  $\mathcal{C}$  attains value  $1 \neq r, 0$ , and (36) leads to unequivocal inequalities.

From this theorem we see that the above comparison with crisp complement does not suffice for introduction better than ordinary complement of fuzzy sets. In this situation we have the two ways: a) considering a weaker interpretation of the Definition, which leads to inequality in (35) and to the inclusion in (36); b) considering connections with early chosen union and intersection (cf. e.g. Weber [7]).

Using Lemma 2' in the same manner as Lemma 2 was used in Theorem 1 we get

Theorem 3. Let

$$(37) \quad \phi(T)(x) = \phi(\mathbb{U}_X) \text{ for } T \subset L(X), \mathbb{U}_X = \{A(x) \mid A \in T\}, x \in X,$$

where  $\phi : 2^L \rightarrow L$  is given.

If L is infinite then:

1°  $\phi$  is an ordinary infinite union in  $L(X)$  iff

$$(38) \quad \phi(\{0\}) = 0, \quad \phi(\{0,1\}) = \phi(\{1\}) = 1,$$

and it is a singular union iff (38) does not hold.

2°  $\phi$  is an ordinary infinite intersection in  $L(X)$  iff

$$(39) \quad \phi(\{0\}) = \phi(\{0,1\}) = 0, \quad \phi(\{1\}) = 1,$$

and it is a singular intersection iff (39) does not hold.

3°  $\phi$  is a uniform infinite union in  $L(X)$  iff

$$(40) \quad \phi(\{0\}) = 0, \quad \phi(\{0,r\}) = \phi(\{r\}) = r \text{ for } r \in L.$$

4°  $\phi$  is a uniform infinite intersection in  $L(X)$  iff

$$(41) \quad \phi(\{0\}) = \phi(\{0,r\}) = 0, \quad \phi(\{r\}) = r \text{ for } r \in L.$$

5°  $\phi$  is a regular ( $g$ -regular) infinite union in  $L(X)$  iff L is a chain with property (\*) and (12') holds.

6°  $\phi$  is a regular ( $g$ -regular) infinite intersection in  $L(X)$  iff L is a complete lattice and (11') holds.

7° If L is a chain, then the unique s-regular infinite union is defined by (12'), and the unique s-regular infinite intersection is defined by (11') iff L has property (\*\*).

After Examples 2 and 3 we see that s-regular operations are not unique if L is not a chain.

Let  $n \in \mathbb{N}$  and  $\phi : L^n \rightarrow L$ . Now we ask whether the formula

$$(42) \quad (A_1 \times \dots \times A_n)(x_1, \dots, x_n) = \phi(A_1(x_1), \dots, A_n(x_n))$$

for  $A_i \in L(X_i)$ ,  $x_i \in X_i$ ,  $i = 1, 2, \dots, n$ , defines a Cartesian product of sunny sets. Repeating a part of the proof of Theorem 1 we get

Theorem 4. Let  $C = A_1 \times \dots \times A_n$  from (42).

1° C is an ordinary Cartesian product iff

$$(43) \quad \phi(a) \in \{0,1\} \text{ for } a \in \{0,1\}^n \text{ and } \phi(a) = 1 \Leftrightarrow \bigvee_i (a_i = 1),$$

and it is a singular Cartesian product iff (43) does not hold.

2° C is a uniform Cartesian product iff

$$(44) \quad \phi(a) \in \{0,r\} \text{ for } a \in \{0,r\}^n \text{ and } \phi(a) = r \Leftrightarrow \bigvee_i (a_i = r) \text{ for } r \in L.$$

3° C is a regular Cartesian product iff (11) holds.

4° If L is a chain, then C is an s-regular cartesian product iff (11) holds.

5° The g-regular Cartesian product does not exist (Cartesian product of endograph has another dimension then endograph of Cartesian product).

A comparison between 5° - 7° in Theorems 1 and 3, and between 3° - 5° in Theorem 4 shows that different kinds of regularity are not comparable.

Using Lemma 2' and 2'' we can formulate the following theorems for arbitrary and for countable Cartesian product:

Theorem 4'. Let  $\phi : 2^L \rightarrow L$ ,  $A_s \in L(X_s)$  for  $s \in S$ , and

$$(45) \quad (\underset{s \in S}{\times} A_s)(x) = \phi(\{A_s(x_s) | s \in S\}) \text{ for } x = (x_s)_{s \in S} \in \underset{s \in S}{\times} X_s.$$

If  $S$  and  $X$  are infinite then we get the following general Cartesian products:  
 1° (45) is an ordinary Cartesian product iff (39) holds and it is a singular Cartesian product iff (39) does not hold.

2° (45) is a uniform Cartesian product iff (41) holds.

3° (45) is a regular Cartesian product iff  $L$  is a complete lattice and (11') holds.

4° If  $L$  is a chain, then (45) is an  $s$ -regular Cartesian product iff  $L$  has property (\*\*) and (11') holds.

Theorem 4''. Let  $\phi : L^{\mathbb{N}} \rightarrow L$ ,  $x = (x_1, x_2, \dots)$ , and

$$(46) \quad (\underset{i \in \mathbb{N}}{\times} A_i)(x) = \phi(A_1(x_1), A_2(x_2), \dots) \text{ for } A_i \in L(X_i), x_i \in X_i, i \in \mathbb{N}.$$

If  $L$  is infinite, then we get the following countable Cartesian products:

1° (46) is an ordinary Cartesian product iff

$$(47) \quad \phi(a) \in \{0, 1\} \text{ for } a \in \{0, 1\}^{\mathbb{N}} \text{ and } \phi(a) = 1 \Leftrightarrow \bigvee_i (a_i = 1)$$

and it is a singular Cartesian product iff (47) does not hold.

2° (46) is a uniform Cartesian product iff

$$(48) \quad \phi(a) \in \{0, r\} \text{ for } a \in \{0, r\}^{\mathbb{N}} \text{ and } \phi(a) = r \Leftrightarrow \bigvee_i (a_i = r) \text{ for } r \in L.$$

3° (46) is a regular Cartesian product iff  $L$  is meet- $\mathfrak{S}_1$ -complete and (11'') holds.

4° If  $L$  is a chain, then (46) is an  $s$ -regular Cartesian product iff  $L$  has property (\*\*) and (11'') holds.

The above theorems are connected with so called "extension principle" (cf.

Zadeh [10] or Dubois, Prade [6]). A function  $f : X_1 \times \dots \times X_n \rightarrow Y$  is extended to  $f : L(X_1) \times \dots \times L(X_n) \rightarrow L(Y)$  by the formula

$$(50) \quad f(A_1, \dots, A_n)(y) = \bigvee_{y=f(x_1, \dots, x_n)} (A_1 \times \dots \times A_n)(x_1, \dots, x_n) \text{ for } y \in Y,$$

where the supremum is taken on the set

$$\{(x_1, \dots, x_n) \in X_1 \times \dots \times X_n | f(x_1, \dots, x_n) = y\}.$$

For  $n = 1$  we have the simplest case

$$(50) \quad f(A)(y) = \bigvee_{y=f(x)} A(x) \text{ for } y \in Y, A \in L(X) \text{ and } f : X \rightarrow Y.$$

After Theorems 1, 3 and 4 we get

Theorem 5. Let  $L$  be a complete lattice. Then (50) is a uniform function image (for fuzzy sets). Moreover, if  $L$  is a chain, then

1° (50) is the unique  $s$ -regular function image.

2° (50) is the unique regular ( $\mathfrak{S}$ -regular) function image iff  $L$  has property (\*\*) or  $L$  is finite or  $X$  is finite.

Theorem 6. Let  $L$  be a complete lattice. If the Cartesian product in (49) is singular, ordinary or uniform, then (49) gives a singular, ordinary or uniform extension of function  $f$ , respectively.

Moreover, if  $L$  is a chain and the Cartesian product in (49) is defined by (42) with (11), then

1° (49) is an s-regular extension of function  $f$ .

2° If  $L$  has property (\*\*) or  $L$  is finite or  $X_1 \times \dots \times X_n$  is finite, then (49) is a regular extension of function  $f$ .

In application of the above theorem leads to the consideration of ordinary, uniform or regular calculus of fuzzy numbers (for  $X = \mathbb{R}$  and  $f: X \times X \rightarrow X$  defined by arithmetic operations; cf. e.g. Dubois, Prade [6], Chapter 2).

#### References

- [1] G.Birkhoff, Lattice theory, AMS Coll.Publ.25, 3<sup>rd</sup> ed., New York 1967.
- [2] L.Czogała, J.Drewniak, Associative monotonic operations in fuzzy set Theory, Fuzzy Sets Syst.12(1984) 249-269.
- [3] J.Drewniak, Axiomatic systems in fuzzy algebra, Acta Cybernet.5(1981) 181-206.
- [4] \_\_\_, Calculus of fuzzy sets I, BUSEFAL 21(1985)
- [5] \_\_\_, Fuzzy relation calculus (to appear).
- [6] B.Dubois, H.Prade, Fuzzy sets and systems, Acad.Press, New York 1980.
- [7] T.Radecki, Level fuzzy sets, J.Cybernet.7(1977) 189-198.
- [8] C.V.Negoita, D.A.Ralescu, Applications of fuzzy sets to system analysis, Birkhäuser, Basel 1975.
- [9] E.Neber, A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms, Fuzzy Sets Syst.11(1983) 115-134.
- [10] Z.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Amer.Elsevier, New York 1973.