The Axiomatics of Pan-Boolean Algebra

Zhu Wujia* Zhang Nanlun** Xiao Xian***

*Dept. of Mathematics, Nanjing University, China

**Dept. of Automation, Wuhan Institute of Building Materials, China

***The Group of Mathematics, Meteorology College of the Air-force, China

Summary

If some practical problem cotains variables $X_1, X_2, \ldots$ and each variable state may be $n_1, n_2, \ldots$ $(2 \leq n_i < \omega, i=1,2,\ldots)$, then it is said that the practical problem constitutes a multiple-state system. In this paper, the axiomatics of pan-boolean algebra has been established, thus a method of logical analysis dealing with multiple-state system (for example, fuzzy models of static) is provided.

The axiomatics is as follows:

1. Fundamental symbols

(1) variable symbols: $X_1, X_2, \ldots$;

(2) state number symbols: $n_1, n_2, \ldots (n_i \geq 2,\ i=1,2,\ldots)$,

(3) state variable symbols: $x_i^1, x_i^2, \ldots, x_i^{n_i} (i=1,2,\ldots)$,

(4) operation symbols: + (logical sum), $\cdot$ (logical product),

(5) constant symbols: 0, 1;

(6) logical predicate: =,

(7) technical symbol: (,).

2. The rules of formation

(1) Singular constant symbol or state variable symbol is pan-boolean expression.

(2) If A and B are pan-boolean expressions, then $(A+B)$ and $(A\cdot B)$ are also pan-boolean expressions (The exteriorest brackets may be omitted).

(3) All pan-boolean expressions can be produced only by (1), (2).

3. Axioms

There are five axioms in pan-boolean algebra (PB).

A_1 (Commutative laws):

(1) $A+B=B+A$,

(2) $A\cdot B=B\cdot A$;

A_2 (Distributive laws):

(1) $A\cdot(B+C)=A\cdot B+A\cdot C$,

(2) $A+(B\cdot C)=(A+B)\cdot(A+C)$;

A_3 (Identity laws):

(1) $A+0=A$,

(2) $A\cdot 1=A$;

A_4 (Associative laws):

(1) $(A+B)+C=A+(B+C)$,

(2) $(A\cdot B)\cdot C=A\cdot(B\cdot C)$;

A_5 (Laws of state):

(1) For every variable X_i, there exists one and only one n_i;

(2) $x_i^j\cdot x_i^k=0$, $(1\leq j<k\leq n_i)$;

(3) $x_i^1+x_i^2+\ldots+x_i^{n_i}=1$.

The operations of complement in PB can be defined recusively as follows:

Definitions

(1) $\overline{0}=df\ 1$,

$\overline{1}$=df 0,

$\overline{x_i^j}$=df $x_i^1+x_i^2+\ldots+x_i^{j-1}+x_i^{j+1}+\ldots+x_i^{n_i}$ ($1\leq j\leq n_i$)

(2) If the complements of A and B are $\overline{A}$ and $\overline{B}$ respectively, then $\overline{(A+B)}$ =df$(\overline{A}\cdot\overline{B})$, $(\overline{A\cdot B})$ =df$(\overline{A}+\overline{B})$.

By using a series of lemmas, we can prove the laws of complementarity in PB:

Theorem For any pan-boolean expression A, we have

$A+\overline{A}=1$ and $A\cdot\overline{A}=0$.

Therefore, any formal theorem in boolean algebra can also be proved in pan-boolean algebra, except that $\overline{A}$ has different contents.