DECOMPOSITION OF SOFT ALGEBRA

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ABSTRACTION

In this paper we will give a decomposition of soft algebra, then, by means of this result, we will give a definition of degree of soft algebra which extends the conception of measure of fuzziness.

1. INTRODUCTION AND DEFINITION

Fuzzy phenomena exist everywhere in the world , what is the universal law of fuzziness? To make out this question , we will do well to manage to translate fuzziness into certainty. As have been shown, we can use λ-cut to realize this purpose. But, only with λ-cut, we can not grasp the essence of fuzziness completely, some direct relationships between fuzziness and certainty are missed. So we want to consider the relationships in other way.

A partial ordered set L is called a upper semilattice if for all α, β in L, there is a least upper bound in L; A partial ordered set is called a lower semilattice if for all α, β in L, there is a great lower bound in L.

A partial ordered set L is called a lattice if L is both a upper semilattice and a lower semilattice.

A lattice is called a dual lattice if there is a operator **c** on L, such that:

1) $(\alpha^c)^c = \alpha \qquad \forall \alpha \in L;$

2) $(\alpha \vee \beta)^c = \alpha^c \wedge \beta^c \qquad (\alpha \wedge \beta)^c = \alpha^c \vee \beta^c$ L

A lattice is called a distributive lattice if the operation $\vee$ and $\wedge$ satisfy the condition $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ for all α, β and γ in L.

A lattice is called a soft algebra if it is a dual distributive lattice.

2. LATTICE DECOMPOSABLE MAPPING AND DIVISION OF SOFT ALGEBRA

In this section we will give a conception of lattice decomposable mapping, then, we will show that each such a mapping can introduce a division of L. first, we give

Definition 2.1 Let L be a soft algebra. If f is a mapping from L to L satisfying following conditions:

L.1) $f(\alpha)\leqslant\alpha$, $\forall\alpha\in L$;

L.2) $(\alpha\wedge\beta)=(f(\alpha)\wedge\beta)\vee(f(\beta)\wedge\alpha)$, $\forall\alpha,\beta\in L$.

then we call f a lower semilattice decomposable mapping.

For a mapping f from L to L, we can introduce a relation $\sim$ in L, such that $\alpha\sim\beta$ iff $f(\alpha)=f(\beta)$. It is easy to see $\sim$ to be a equivalent relation. Let $L/f=\{E_f[\alpha]\}$, where $E_f[\alpha]=\{\beta|\ f(\beta)=f(\alpha)\}$.

From above definition, we can obtain the following proposition immediatly.

Proposition2.1 Let f be a lower semilattice decomposable mapping of soft algebra L, then each element of L/f makes up a lower semilattice in the induced order.

Proof For all β, γ in $E_f[\alpha]$, then, $f(\beta)=f(\gamma)=f(\alpha)$, by L.1), L.2), we have

$$\begin{aligned}f(\beta\wedge\gamma)&=(f(\beta)\wedge\gamma)\vee(f(\gamma)\wedge\beta)=(f(\gamma)\wedge\gamma)\vee(f(\beta)\wedge\beta)\\&=f(\gamma)\wedge f(\beta)=f(\alpha)\end{aligned}$$

therefore, $\beta\wedge\gamma\in E_f[\alpha]$. i.e. $E_f[\alpha]$ closes for operation $\wedge$. This is to say $E_f[\alpha]$ is a lower semilattice in the induced order.

Definition 2.2 Let L be a soft algebra, f be a mapping from L to L. f is called a upper semilattice decomposable mapping, if f satisfies the conditions:

L.3) $f(\alpha)\leqslant\alpha^c$, $\forall\alpha\in L$;

L.4) $f(\alpha\vee\beta)=(f(\alpha)\wedge\beta^c)\vee(f(\beta)\wedge\alpha^c)$ $\forall\alpha,\beta\in L$.

Similar with proposition2.1, we also have

Proposition 2.2 Let L be a soft algebra, f be a upper semilattice decomposable mapping of L. Then each element of $L/f=\{E_f[\alpha]\}$ makes up a upper semilattice in the induced order.

Definition 2.3 Let L be a soft algebra, a mapping f from L to L is said to be a lattice decomposable mapping, if f satisfies L.1)—L.4).

It is easy to see f is a lattic decomposable mapping of lattice L iff f is both lower semilattice decomposable mapping and upper semilattice decomposable mapping.

Following proposition is evident.

Proposition 2.3 If f is a lattice decomposable mapping of a soft algebra L, then for all α, $\beta\in L$,

1) $f(\alpha\vee\alpha^c)=f(\alpha\wedge\alpha^c)=f(\alpha)\vee f(\alpha^c)$;

2) $f(\alpha\vee\beta)\leqslant(f(\alpha)\vee f(\beta))\wedge(\alpha\wedge\beta)^c$,

$f(\alpha\wedge\beta)\leqslant(f(\alpha)\vee f(\beta))\wedge(\alpha\wedge\beta)$;

3) $f(\alpha\vee\beta)\vee f(\alpha\wedge\beta)=(f(\alpha)\wedge(\beta\vee\beta^c))\vee(f(\beta)\wedge(\alpha\vee\alpha^c))$.

Proposition 2.4 If f is a lattice decomposable mapping of a soft algebra L, then each element of L/f makes up a sublattice in the induced order.

This is simply a corollary of proposition 2.1 and 2.2 .

Definition 2.4 A lattice decomposable mapping f is said to be dual, if

L.5) $f(\alpha) = f(\alpha^c) \quad \forall \alpha \in L$.

holds.

From the definition of dual lattice decomposable mapping, we have

Proposition 2.5 Let L be a soft algebra, and f be a dual lattice decomposable mapping of L. Then for each $E_f[\alpha]$ in L/f , $E_f[\alpha]$ makes up a dual sublattice in the induced order and induced dual operation.

From the discussion above, we have seen that every lattice decomposable mapping of a lattice L not only brings us a division of L, but also keeps each member of L/f is a sublattice of L; If L is complete, then each member of L/f is complete; If f is dual, then each member of L/f is a dual sublattice of L. In the following part of this paper , we should only consider a very particular lattice decomposable mapping——boolean decomposable mapping f_0 , That is for all α in L, $f_0(\alpha) = \alpha \wedge \alpha^c$.

It is easy to see that f_0 satisfies L.1)——L.5). Therefore f_0 is a dual lattice decomposable mapping. By the proposition 2.4, every element of L/f_0 is a dual sublattice of L in the induced order. But with the definition of f_0 , we know that for all β in $E_{f_0}[\alpha]$, $\beta \wedge \beta^c = \alpha \wedge \alpha^c$, $\beta \vee \beta^c = \alpha \vee \alpha^c$, so ($E_{f_0}[\alpha]$, $\leq$) is a boolean algebra with $0_\alpha = \alpha \wedge \alpha^c$ zero member and $1_\alpha = \alpha \vee \alpha^c$ unit member. therefore, we get

Theorem 2.1 Every soft algebra has a division such that each part of the division makes up a boolean algebra in the induced order.

Definition 2.5 Let L be a complete dual distrbutive lattice. Let $0 = \bigwedge L$, $1 = \bigvee L$, we call the element in $E_{f_0}[0]$ to be sharpen element of L.

Let X be a ordinary set, F(X) is consists of all fuzzy sets of X and P(X) is consists of all ordinary sets of X. Because F(X) make up a soft algebra in the ordinary order, therefore, we get

Corollary 2.1 Let $L_1 = F(X)$, $L_0 = P(X)$, if f is a dual lattice decomposable mapping on L, then $E_f[0]$ Is a dual fuzzy sublattice of F(X) with 0 the smallest member and 1 the greatest member. Particularly for f_0 , $E_{f_0}[0] = L_0 = P(X)$.

By means of theorem 2.1, we can write out the form of the number of L when L is finit.

Corollary 2.2 If L is a finit soft algebra , then $|L| = 2^{n_\alpha}$, where $|L|$ denote the number of element in L, and 2^{n_α} is the number of element in $E_{f_0}[\alpha]$.

Remark It is not difficulty to show that for any finit number n , there must be a soft algebra L, such that $|L| = n$. But, when supposing no α in L such that $\alpha^c = \alpha$, then $|L| = 2k$ for some natural number k . If we want to express it more precisely, we can write $|L| = \sum s_i 2^i$, (i= 0, 1, 2, ...), where s_i denote the number of $\{E_{f_0}[\alpha] \mid |E_{f_0}[\alpha]| = 2^i\}$.

3. SOME PROPERTY OF $E_{f_0}[\alpha]$

In the first section, we have discussed some kinds of lattice decomposable mapping of a Lattice L. In this section we principally consider the relationship between different members of $\{E_{f_0}[\alpha]\}$.

In the following part of this paper, we always Let L be a soft algebra , f_0 be boolean decomposable mapping from L to L , i.e. for all α in L, $f_0(\alpha) = \alpha \wedge \alpha^c$. Next, instead of $E_{f_0}[\alpha]$, we use $E[\alpha]$ denotes the member of L/f_0 .

Proposition 3.1 $\forall \alpha, \beta \in L$, let $0_\alpha = \alpha \wedge \alpha^c$, $0_\beta = \beta \wedge \beta^c$, and $1_\alpha = \alpha \vee \alpha^c$, $1_\beta = \beta \vee \beta^c$, then

1) If $\alpha \geqslant \beta$ then $\beta \wedge \alpha^c = 0_\alpha \wedge 0_\beta$, $\beta^c \vee \alpha = 1_\alpha \vee 1_\beta$;

2) $\forall \gamma \in E[\alpha \vee \beta]$, $\gamma \wedge \gamma^c \geqslant 0_\alpha \wedge 0_\beta$, $\gamma \vee \gamma^c \leqslant 1_\alpha \vee 1_\beta$, (the same is true for γ in $E[\alpha \wedge \beta]$;

3) If $0_\alpha \wedge \beta^c = 0_\alpha \wedge 0_\beta$, then $0_{\alpha \vee \beta} = 0_\alpha \wedge 0_\beta$, if $\beta \vee 1_\alpha = 1_\alpha \vee 1_\beta$, $\alpha \vee 1_\beta = 1_\alpha \vee 1_\beta$, then $1_{\alpha \vee \beta} = 1_\alpha \vee 1_\beta$;

4) $0_\alpha \leqslant 0_\beta$ iff $1_\alpha \geqslant 1_\beta$,
$0_\alpha = 0_\beta$ iff $1_\alpha = 1_\beta$.

The proof is direct.

Proposition 3.2 Let $\{0_\alpha\} = \{\alpha \wedge \alpha^c \mid \alpha \in L\}$, then $\{0_\alpha\}$ make up a lower semilattice in the induced order. Particularly, if for all α , β in L, $0_\alpha \vee 0_\beta \leqslant 1_\alpha \wedge 1_\beta$, then $\{0_\alpha\}$ make up a lattice in the induced order.

The proof is evident.

Definition 3.1 Soft algebra L is called local boolean , if $\{0_\alpha\}$ make up a lattice in the induced order.

Proposition 3.3 F(X) is local boolean . Furthermore, for all A in F(X), if $A(x) \neq 1/2$ for all x in X, then $E[A]$ has the same cardinal with $E[0] = P(X)$.

From the discussion above , we can see that soft algebra is nothing but a mapping with a domain semilattice and a range collection of boolean algebras . In other words, a soft algebra can be regarded as a plate of lattice on which every point endows with a boolean algebra. this idea suggests that fuzzy computor can be regarded as a group of ordinary computors which are arranged in certain partial order. In following part of this paper , we will mainly discuss under what condition can a group of boolean algebras decide a soft algebra.

For each $E[\alpha]$, let $\hat{E}[\alpha]=\{\gamma \mid 0_\alpha \leqslant \gamma \leqslant 1_\alpha,\ \gamma \in L\}$. It is easy to see that $\hat{E}[\alpha]$ is a dual sublattice . Evidently, $E[\alpha]$ is consists of all sharpen elements of $\hat{E}[\alpha]$. We call $E[\alpha]$ to be frame of $\hat{E}[\alpha]$, and noted as $E[\alpha]=F_r(\hat{E}[\alpha])$.

Definition 3.2 A sublattice of a soft algebra M is said to be substantial, If for all α, β in M, $\alpha \leqslant \beta$, then, $\forall\ \gamma$, $\alpha \leqslant \gamma \leqslant \beta$ we have $\gamma \in M$.

Obviously, $\hat{E}[\alpha]$ is substantial for all $E[\alpha]$ in L/f_0 .

Proposition 3.4 Let $E[\alpha]$, $E[\beta] \in L/f_0$, then

1) If $0_\alpha \leqslant 0_\beta$, then $\hat{E}[\beta]$ is a substantial dual lattice of $\hat{E}[\alpha]$;
2) Both $\hat{E}[\alpha]$ and $\hat{E}[\beta]$ is substantial dual sublattice of $\hat{E}[0_\alpha \wedge 0_\beta]$;
3) $\hat{E}[\alpha] \cap \hat{E}[\beta]$ is the biggest substantial dual sublattice containing in $\hat{E}[\alpha]$ and $\hat{E}[\beta]$.

The proof is simple.

Definition 3.3 Let T be a complete lower semilattice $\{L_t\}_{t \in T}$ is a collection of soft algebras. A collection $\{f_{(t_1,t_2)} \mid \forall\ t_1,t_2 \in T,\ f_{(t_1,t_2)}$ is a mapping from $L_{t_1} \cup L_{t_2}$ to $L_{t_1 \wedge t_2}\}$ is called a interrelated mapping collection on T, If for all α in L_{t_1} ,

$$f_{(t_1 \wedge t_2,\ t_3)} \circ f_{(t_1,t_2)}(\alpha) = f_{(t_1 \wedge t_3,\ t_2)} \circ f_{(t_1,t_3)}(\alpha) \quad \text{for all } t_1,\ t_2,\ t_3 .$$

Definition 3.4 Let T be a complete lower semilattice , a collection of soft algebras $\{L_t\}_{t \in T}$ is called auto-agree, if there exists a interrelated mapping collection F on T, such that for all $f_{(t_1,t_2)}$ in F

(1) Under $f_{(t_1,t_2)}$, $L_{t_1} \cong f_{(t_1,t_2)}(L_{t_1})$ and $L_{t_2} \cong f_{(t_1,t_2)}(L_{t_2})$,

where $\cong$ denotes isomorphism between lattices;

(2) If $t_1 \leqslant t_2$, then $f_{(t_1,t_2)}(L_{t_2})$ is a substantial dual sublattice of L_{t_1} .

(3) If $t_1 = t_2$, then $F_r(f_{(t_1,t_2)}(L_{t_1})) \cap F_r(f_{(t_1,t_2)}(L_{t_2})) = 0$;

(4) If 0_{t_1}, 0_{t_2} is the minimum member of L_{t_1}, L_{t_2} respectively, then

$$f_{(t_1,t_2)}(0_{t_1}) \wedge f_{(t_1,t_2)}(0_{t_2}) = 0_{t_1 \wedge t_2} .$$

Definition 3.5 Let T be a complete lower semilattice, a collection of boolean algebras $\{E_t\}_{t \in T}$ is called suitable with a given auto-agree collection of soft algebras $\{L_t\}_{t \in T}$ if for any $t \in T$, there exists a isomorphism between E_t and $F_r(L_t)$.

Theorem 3.1 Let T be a lower semilattice, $\{L_t\}(t \in T)$ is a auto-agree collection of soft algebras, collection of boolean algebras $\{E_t\}(t \in T)$ is suitable with $\{L_t\}(t \in T)$. then, there must exists a soft algebra L such that $E[\alpha] \in L/f_0$ iff there is t in T, such that $E[\alpha]$ isomorphies to E_t .

The proof is omitted.

4. EXTENDED MEASURE OF FUZZINESS

The degree of fuzziness is assumed to express on a global level the difficulty Of deciding which elements belong and which do not belong to a given fuzzy sets.

Mathematically, as D.luca and Tuemini has proposed , a measure of fuzziness is a mapping **d** from F(X) to $[0, +\infty)$ satisfying conditions

(1) $\mathbf{d}(A) = 0$ iff A is a ordinery subset of X;

(2) $\mathbf{d}(A)$ is maximum iff $\mu_A(x)=1/2 \quad \forall\ x \in X$;

(3) $\mathbf{d}(A^*) \leqslant \mathbf{d}(A)$, where A^* is any sharpen version of A, that is , $\mu_{A^*}(x) \leqslant \mu_A(x)$ if $\mu_A(x) \leqslant 1/2$ and $\mu_{A^*}(x) \geqslant \mu_A(x)$ if $\mu_A(x) \geqslant 1/2$;

(4) $\mathbf{d}(A) = \mathbf{d}(\bar{A})$, When $\bar{A}$ is as fuzzy as A .

By theorem 2.1 , it is reasonable to regard that that $\bar{A}$ is as fuzzy as A can be discribed as $\bar{A} \wedge \bar{A}^c = A \wedge A^c$. Therefore , in order to extend above definition of degree of fuzziness to soft algebra, we give following definition of degree of soft algebra.

Definition 4.1 A mapping **d** from a soft algebra L to $[0, +\infty)$ is said to be a degree of soft algebra , if **d** satisfies conditions

d.1) If $\alpha \wedge \alpha^c \leqslant \beta \wedge \beta^c$, then $\mathbf{d}(\alpha) \leqslant \mathbf{d}(\beta) \quad \forall\ \alpha, \beta \in L$;

d.2) $\mathbf{d}(\alpha)=\mathbf{d}(\beta)$ iff $\alpha \wedge \alpha^c = \beta \wedge \beta^c \quad \forall \alpha, \beta \in L$;

d.3) $\mathbf{d}(\alpha)=0$ iff $\alpha \in \mathbf{E}[0]$.

Evidently, we have

Proposition 4.1 Let L be a soft algebra , **d** is a degree of L,

(1) If $\alpha = \alpha^c$, then , for all β in L, $\beta \leq \alpha$, we have $\mathbf{d}(\beta) \leqslant \mathbf{d}(\alpha)$;

(2) $\mathbf{d} - \mathbf{d}(\alpha)$ is a degree of $\hat{\mathbf{E}}[\alpha]$.

The proof is direct.

When we restrict L to be F(X), the degree of F(X) is nothing but the measure of fuzziness if only we pay attention to the fact that for any A in F(X), that $A \wedge A^c \leqslant 1/2$ is always true.

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