THE CONVERGENCES IN EIGHT KINDS OF HYPERTOPOLOGIES AND THEIR APPLICATIONS

Zhang Xing-hu & Wang Pei-zhuang
(Dept.of Math., Beijing Normal University, Beijing, China)

In literatures [1],[2],Wang has introduced eight kinds of hypertopologies through lattice topologies.These eight kinds of hypertopologies include the hypertopologies provided in [3],[4],[5].Moreover,they include the several new hypertopologies not defined before.In this paper,taking convergences of set-net in base space as tool,we give the laws of convergence for each of eight hypertopologies,and give the rigion limit points of set-net in,and make a comparison between all kinds of hypertopologies.Extraordinarily,in collection $\mathcal{H}$ of all clopen subsets of X,the comparison between the eight kinds of hypertopologies has a good result.Using the convergence-tool,we will discuss the coincidence of the product space of the hyperspaces deduced from X,Y and the hyperspace of the product space of them.Finally,using convergence-tool,we will discuss the relations between the hypertopologies in X and the order topology,the Scott topology in power set lattice $\mathcal{P}(X)$.

Keywords: Hypertopology, Hyperspace, Set-net, Convergence, Comparision between hypertopologies, Product of hypertopologies.

In this paper, we adopt essentially the notations in [1],[2]. For convenience later, we begin by listing some of our pricipal conventions and notations:

CONVENTIONS:

(1) A set-net $\{A_d\}_{d\in D}$ in X is a net in the power set $\mathcal{P}(X)$ of X;

(2) $\subset$ indicates the set-theoretic inclusion, $\subsetneq$ indicats the proper inclusion.

(3) Let $(X,\mathcal{G})$ be a topological space, $\mathcal{P}(X)$ denotes the power set of X. ${}^{\circ}\mathcal{P}(X)\triangleq\mathcal{P}(X)\setminus\{X\}$, ${}_{\circ}\mathcal{P}(X)\triangleq\mathcal{P}(X)\setminus\{\phi\}$, ${}^{\circ}_{\circ}\mathcal{P}(X)\triangleq\mathcal{P}(X)\setminus\{\phi,X\}$. $\mathcal{F}(X)$ denotes the collection of all closed subsets of X, $\mathcal{G}(X)$ denotes the collection of all open subsets of X, $\mathcal{H}(X)$ denotes the collection of all clopen subsets of X.F ranges over closed subsets of X,G ranges over open subsets of X,and H ranges over clopen subsets of X.

(4) For each $A\in\mathcal{P}(X)$, let $\dot{A}=\{B\in\mathcal{P}(X)\mid B\supset A\}$,and $\underset{\cdot}{A}=\{B\in\mathcal{P}(X)\mid B\subset A\}$,and for each $\mathcal{D}\subset\mathcal{P}(X)$, let $\mathcal{D}^{\backslash c}=\{A^c=X\setminus A\mid A\in\mathcal{D}\}$.

(5) There are definitions as follow in literatures [1],[2]:

$T_{10}(\mathcal{P}(X))$ denotes the hypertopology in X which base is $\dot{\mathcal{G}} \triangleq \{\dot{G} | G \in \mathcal{G}\}$. That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{10})$ will be denoted as $A_d \longrightarrow A$, and will be called convergence.

$T_{01}(\mathcal{P}(X))$ denotes the hypertopology in X which base is $\dot{\mathcal{F}} \triangleq \{\dot{F} | F \in \mathcal{F}\}$. that $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{01}(\mathcal{P}(X))$ will be denoted as $A_d \longrightarrow A$, and will be called anti-convergence.

$T_{20}(\mathcal{P}(X))$ denotes the hypertopology in X which subbase is $(\dot{\mathcal{G}})^c \triangleq \{\mathcal{P}(X)\setminus \dot{G} | G \in \mathcal{G}\} \cup \{\mathcal{P}(X)\}$. That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{20}(\mathcal{P}(X))$ will be denoted as $A_d \twoheadrightarrow A$, and will be called pan-convergence.

$T_{02}(\mathcal{P}(X))$ denotes the hypertopology in X which subbase is $(\dot{\mathcal{F}})^c \triangleq \{\mathcal{P}(X)\setminus \dot{F} | F \in \mathcal{F}\} \cup \{\mathcal{P}(X)\}$. That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{02}(\mathcal{P}(X))$ will be denoted as $A_d \twoheadrightarrow A$, and will be called anti-pan-convergence.

According to the four kinds of hypertopologies above, we can obtain four kinds of hypertopologies as follow:

$T_{11}(\mathcal{P}(X))$ denotes the hypertopology in X which subbase is $\dot{\mathcal{G}} \cup \dot{\mathcal{F}}$. That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{11}(\mathcal{P}(X))$ will be denoted as $A_d \longrightarrow A$.

$T_{12}(\mathcal{P}(X))$ denotes the hypertopology in X which subbase is $\dot{\mathcal{G}} \cup (\dot{\mathcal{F}})^c$. That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{12}(\mathcal{P}(X))$ will be denoted as $A_d \twoheadrightarrow A$.

$T_{21}(\mathcal{P}(X))$ denotes the hypertopology in X which subbase is $(\dot{\mathcal{G}})^c \cup (\dot{\mathcal{F}})^c$. That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{21}(\mathcal{P}(X))$ will be denoted as $A_d \twoheadrightarrow A$.

$T_{22}(\mathcal{P}(X))$ denotes the hypertopology in X which subbase is $(\dot{\mathcal{G}})^c \cup (\dot{\mathcal{F}})^c$ That $\{A_d\}_{d\in D}$ converge to A in $(\mathcal{P}(X), T_{22}(\mathcal{P}(X))$ will be denoted as $A_d \twoheadrightarrow A$.

(6) Let $\mathcal{D} \subset \mathcal{P}(X)$, $T_\omega(\mathcal{D})$ indicates the deduced topology on $\mathcal{D}$ of $T_\omega(\mathcal{P}(X))$, where $\omega \in \{10, 01, 20, 02, 11, 12, 21, 22\}$.

(7) $T_\omega(\mathcal{D})$ denotes the space $(\mathcal{D}, T_\omega(\mathcal{D}))$, and sametime denotes the topology $T_\omega(\mathcal{D})$.

§1. Basic Definitions

Definition 1.1 Let $\{A_d\}_{d\in D}$ be a set-net in topological space $(X, \mathcal{G})$. Define $\overline{\lim}_D A_d = \{x \in X |$ For every neighbourhood U(x) of x, there exists a subnet $\{A_{d'}\}_{d'\in D'}$ of net $\{A_d\}_{d\in D}$ such that $U(x) \cap A_{d'} \neq \phi$ for every $d' \in D'\}$.

$\underline{\lim}_D A_d = \{x \in X |$ for every neighbourhood U(x) of x, there exists $d_0 \in D$ such that $U(x) \cap A_d = \phi$ for every $d \geq d_0\}$.

$\overline{\overline{\lim}}_D A_d = \{x \in X |$ There exists a neighbourhood U(x) of x and a subnet $\{A_{d'}\}_{d'\in D'}$ such that $U(x) \subset A_{d'}$, for every $d' \in D'\}$.

$\underline{\underline{\lim}}_D A_d = \{x \in X |$ therehexists a neighbourhood U(x) of x and $d_0 \in D$ such that $U(x) \subset A_d$ for every $d \geq d_0\}$.

PROPOSITION 1.1 Let $\{A_d\}_{d\in D}$ be a set-net in X.Then

(1) $\underline{\underline{\lim}}_D A_d \subset \underline{\lim}_D A_d$, (2) $\overline{\overline{\lim}}_D A_d \subset \overline{\lim}_D A_d$,

(3) $\underline{\lim}_D A_d \subset \overline{\lim}_D A_d$, (4) $\underline{\underline{\lim}}_D A_d \subset \overline{\overline{\lim}}_D A_d$,

(5) $\underline{\lim}_D A_d$ $\underline{\lim}_D A_d$ and $\overline{\overline{\lim}}_D A_d$ are not comparable in general.

PROPOSITION 1.2 Let $\{A_d\}_{d\in D}$ be a set-net in X.Then

(1) $\underline{\lim}_D A_d$ and $\overline{\lim}_D A_d$ are always closed in X.

(2) $\underline{\underline{\lim}}_D A_d$ and $\overline{\overline{\lim}}_D A_d$ are always open in X.

PROPOSITION 1.3 If $\{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}$.Then

(1) $\overline{\lim}_{D'} A_{d'} \subset \overline{\lim}_D A_d$, (2) $\overline{\overline{\lim}}_{D'} A_{d'} \subset \overline{\overline{\lim}}_D A_d$,

(3) $\underline{\lim}_{D'} A_{d'} \supset \underline{\lim}_D A_d$, (4) $\underline{\underline{\lim}}_{D'} A_{d'} \supset \underline{\underline{\lim}}_D A_d$.

PROPOSITION 1.4 Let $\{A_d\}_{d\in D}$ be a set-net in X,and $A\in\mathcal{P}(X)$.If there exists $d_0\in D$ such that $A_d \subset A$ for each $d \geqslant d_0$. Then

(1) $\overline{\lim}_D A_d \subset \overline{\lim}_D A = \bar{A}$, $\underline{\lim}_D A_d \subset \underline{\lim}_D A = \bar{A}$;

(2) $\overline{\overline{\lim}}_D A_d \subset \overline{\overline{\lim}}_D A = A^{\circ}$, $\underline{\underline{\lim}}_D A_d \subset \underline{\underline{\lim}}_D A = A^{\circ}$

PROPOSITION 1.5 Let $\{A_d\}_{d\in D}$ be a set-net in X.Then

(1) $\overline{\lim}_D A_d = \bigcup\{\underline{\lim}_{D'} A_{d'} \mid \{A_{d'}\}_{d'\in D}$ is a subnet of $\{A_d\}_{d\in D}\}$.

(2) $\overline{\overline{\lim}}_D A_d = \bigcup\{\underline{\underline{\lim}}_{D'} A_{d'} \mid \{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}\}$.

(3) $\underline{\lim}_D A_d = \bigcap\{\overline{\lim}_{D'} A_{d'} \mid \{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}\}$.

(4) $\underline{\underline{\lim}}_D A_d = \bigcap\{\overline{\overline{\lim}}_{D'} A_{d'} \mid \{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}\}$.

Proof.(1) Taking $x\in\overline{\lim}_D A_d$ (if $\overline{\lim}_D A_d \neq \phi$).We let $K=\{(U,d) \mid U$ is a neighbourhood of $x, d\in D$ and $U\cap A_d \neq \phi\}$.The order in K is defined by $(U_1, d_1) \geqslant (U_2, d_2) \overset{\Delta}{\Longleftrightarrow} U_1 \subset U_2$ and $d_1 \geqslant d_2$.According to the fact that:for every neighbourhood $U(x)$ of x, there exists a subnet $\{A_{d'}\}_{d'\in D}$ of $\{A_d\}_{d\in D}$ such that $U(x)\cap A_{d'} \neq \phi$ ($\forall d'\in D'$),K is a direct set.Let $f: K \longrightarrow D$ be defined by $f((U,d))=d$.Then K is a subdirect set of the direct set D,and $\{A_k\}_{k\in K}$ is a subnet of $\{A_d\}_{d\in D}$ and we have $x\in\underline{\lim}_K A_k$,therefore $\overline{\lim}_D A_d \subset \bigcup\{\underline{\lim}_{D'} A_{d'} \mid \{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}\}$.

$\overline{\lim}_D A_d \supset \bigcup\{\lim_{D'} A_{d'} \mid \{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}\}$ is obvious.

The proofs of (2),(3),(4) are analogous.

§2. The Judging Laws of Net Convergence and the Determinations for the Regions its Limits in

THEOREM 2.1 (The judging laws of limits existing) Let $(X, \mathscr{G})$ be a compact Hausdorff space and $\{A_d\}_{d\in D}$ be a set-net in X. Then

(1) $A_d \rightharpoonup A \Longrightarrow \overline{\lim}_D A_d \subset \bar{A}$; conversely, $\overline{\lim}_D A_d \subset A \Longrightarrow A_d \rightharpoonup A$. In particular, in hyperspace $(\mathscr{F}, T_{10}(\mathscr{F}))$, net $\{F_d\}_{d\in D}$ converge to $F\in\mathscr{F}$, i.e., $F_d \rightharpoonup F \Longleftrightarrow \overline{\lim}_D F_d \subset F$.

(2) $A_d \rightharpoondown A \Longrightarrow \underline{\underline{\lim}}_D A_d \supset A^{\circ}$; conversely, $\underline{\underline{\lim}}_D A_d \supset A \Longrightarrow A_d \rightharpoondown A$. In particular, in hyperspace $(\mathscr{G}, T_{01}(\mathscr{G}))$, net $\{G_d\}_{d\in D}$ converge to $G\in\mathscr{G}$, i.e., $G_d \rightharpoondown G \Longleftrightarrow \underline{\underline{\lim}}_D G_d \supset G$.

(3) $A_d \rightharpoonup\!\!\!\rightharpoonup A \Longleftrightarrow \overline{\overline{\lim}}_D A_d \subset A$.

(4) $A_d \rightharpoondown\!\!\!\rightharpoondown A \Longleftrightarrow \underline{\lim}_D A_d \supset A$.

(5) $A_d \rightarrow A \Longrightarrow \overline{\lim}_D A_d \subset \bar{A}$ and $A^{\circ} \subset \underline{\underline{\lim}}_D A_d$, conversely, $\overline{\lim}_D A_d \subset A \subset \underline{\underline{\lim}}_D A_d$ (therefore $A = \overline{\lim}_D A_d = \underline{\underline{\lim}}_D A_d$) $\Longrightarrow A_d \rightarrow A$. In particular, in hyperspace $(\mathscr{H}, T_{11}(\mathscr{H}))$, net $\{H_d\}_{d\in D}$ converge to $H\in\mathscr{H}$, i.e., $H_d \rightarrow H \Longleftrightarrow H = \overline{\lim}_D H_d = \underline{\underline{\lim}}_D H_d$.

(6) $A_d \twoheadrightarrow A \Longrightarrow \overline{\lim}_D A_d \subset \bar{A}$ and $A \subset \underline{\lim}_D A_d$ (therefore $\bar{A} = \overline{\lim}_D A_d = \underline{\lim}_D A_d$), conversely, $\overline{\lim}_D A_d \subset A \subset \underline{\lim}_D A_d$ (therefore $A = \overline{\lim}_D A_d = \underline{\lim}_D A_d$) $\Longrightarrow A_d \twoheadrightarrow A$. In particularly, in hyperspace $(\mathscr{F}, T_{12}(\mathscr{F}))$, $F_d \twoheadrightarrow F \Longleftrightarrow F = \overline{\lim}_D F_d = \underline{\lim}_D F_d$.

(7) $A_d \rightarrow\!\!\!\rightarrow A \Longrightarrow \overline{\overline{\lim}}_D A_d \subset A$ and $A^{\circ} \subset \underline{\underline{\lim}}_D A_d$ (therefore $A^{\circ} = \underline{\underline{\lim}}_D A_d = \overline{\overline{\lim}}_D A_d$), conversely, $\overline{\overline{\lim}}_D A_d \subset A \subset \underline{\underline{\lim}}_D A_d$ (therefore $A = \overline{\overline{\lim}}_D A_d = \underline{\underline{\lim}}_D A_d$) $\Longrightarrow A_d \rightarrow\!\!\!\rightarrow A$. In particularly, in hyperspace $(\mathscr{G}, T_{21}(\mathscr{G}))$, $G_d \rightarrow\!\!\!\rightarrow G \Longleftrightarrow G = \overline{\overline{\lim}}_D G_d = \underline{\underline{\lim}}_D G_d$.

(8) $A_d \twoheadrightarrow\!\!\!\rightarrow A \Longleftrightarrow \underline{\lim}_D A_d \supset A \supset \overline{\overline{\lim}}_D A_d$.

Remark. The results in (3),(4),(8) always hold for arbitrary topological space $(X, \mathscr{G})$.

Proof. We prove only (1), proofs of the rest of the statements are omitted.

(1) Suppose $A_d \rightharpoonup A$. If $x \notin \bar{A}$, then there exists a neighbourhood $U_0(x)$ of x such that $\bar{U}_0(x) \cap A = \phi$ (because X is regular), i.e., $A \subset \bar{U}_0(x)^c$. Because $A_d \rightharpoonup A$, there exists $d_0\in D$ such that $A_d \subset \bar{U}(x)^c$ for each $d \geqslant d_0$, i.e., $A_d \cap \bar{U}_0(x) = \phi\ (\forall\, d \geqslant d_0)$. Thus, $x \notin \overline{\lim}_D A_d$. Therefore $\bar{A} \supset \overline{\lim}_D A_d$;

Conversely, suppose $\overline{\lim}_D A_d \subset A$. If there exists an open set $U \supset A$, but $\{A_d\}_{d\in D}$ is not eventually included in U. Then there exists a subnet $\{A_{d'}\}_{d'\in D'}$ of $\{A_d\}_{d\in D}$ such that $A_{d'} \not\subset U (\forall d'\in D')$. Let $x_{d'} \in A_{d'} \setminus U$, then $\{x_{d'}\}_{d'\in D'}$ is a net in U^c. Because X is compact, so is U^c. Thus there exists a subnet $\{x_{d''}\}_{d''\in D''}$ of $\{x_{d'}\}_{d'\in D'}$ which converge to $x \in U^c \subset A^c$. Because $\{x_{d''}\}_{d''\in D''}$ converge to $x\in X$, for each neighbourhood U(x) of x there exists $d''_0 \in D''$ such that $x_{d''} \in U(x)$ for every $d'' \geqslant d''_0$, therefore $A_{d''} \cap U(x) \neq \phi$ for every $d'' \geqslant d''_0$. Thus $x \in \overline{\lim}_D A_d \subset A$. It contradicts $x \in A^c$, therefore $A_d \longrightarrow A$.

COLLORY 1. (Determinations for regions limit points in) Let X be a compact Hausdorff space. Then

(1) In hyperspace $(\mathcal{P}(X), \mathbf{T}_{10}(\mathcal{P}(X)))$, the set $\{A\in\mathcal{P}(X) \mid A_d \longrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d\in D}$ is included in the set $\{A\in\mathcal{P}(X) \mid \overline{\lim}_D A_d \subset \bar{A}\}$ and includes the set $\{A\in\mathcal{P}(X) \mid \overline{\lim}_D A_d \subset A\}$, i.e., $\{A\in\mathcal{P}(X) \mid \overline{\lim}_D A_d \subset A\} \subset \{A\in\mathcal{P}(X) \ A_d \longrightarrow A\} \subset \{A\in\mathcal{P}(X) \mid \overline{\lim}_D A_d \subset \bar{A}\}$. In general, the inclusions are proper. But in hyperspace $(\mathcal{F}, \mathbf{T}_{10}(\mathcal{F}))$, the set $\{F\in\mathcal{F} \mid F_d \longrightarrow F\}$ consisting of all limit points of net $\{F_d\}_{d\in D}$ in $\mathcal{F}$ equals the set $\{F\in\mathcal{F} \mid \overline{\lim}_D F_d \subset F\} = (\lim_D F_d)^{\cdot} \cap \mathcal{F}$.

(2) In hyperspace $(\mathcal{P}(X), \mathbf{T}_{01}(\mathcal{P}(X)))$, the set $\{A\in\mathcal{P}(X) \mid A_d \longrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d\in D}$ is included in the set $\{A\in\mathcal{P}(X) \mid \underline{\underline{\lim}}_D A_d \supset A^\circ\}$ and includes the set $\{A\in\mathcal{P}(X) \mid \underline{\underline{\lim}}_D A_d \supset A\}$, i.e., $\{A\in\mathcal{P}(X) \mid \underline{\underline{\lim}}_D A_d \supset A\} \subset \{A\in\mathcal{P}(X) \mid A_d \longrightarrow A\} \subset \{A\in\mathcal{P}(X) \mid \underline{\underline{\lim}}_D A_d \supset A^\circ\}$. In general, the inclusions are proper. But in hyperspace $(\mathcal{G}, \mathbf{T}_{01}(\mathcal{G}))$, the set $\{G\in\mathcal{G} \mid G_d \longrightarrow G\}$ consisting of all limit points of net $\{G_d\}_{d\in D}$ equals the set $\{G\in\mathcal{G} \mid \underline{\underline{\lim}}_D G_d \supset G\} = (\underline{\underline{\lim}}_D G_d)_{\cdot} \cap \mathcal{G}$.

(3) In hyperspace $(\mathcal{P}(X), \mathbf{T}_{20}(\mathcal{P}(X)))$, the set $\{A\in\mathcal{P}(X) \mid A_d \longrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d\in D}$ equals the set $\{A\in\mathcal{P}(X) \mid A \supset \overline{\overline{\lim}}_D A_d\} = (\lim_D A_d)^{\cdot}$.

(4) In hyperspace $(\mathcal{P}(X), \mathbf{T}_{02}(\mathcal{P}(X)))$, the set $\{A\in\mathcal{P}(X) \mid A_d \longrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d\in D}$ equals the set $\{A\in\mathcal{P}(X) \mid A \subset \underline{\lim}_D A_d\} = (\underline{\lim}_D A_d)_{\cdot}$.

(5) In hyperspace $(\mathcal{P}(X), \mathbf{T}_{11}(\mathcal{P}(X)))$, the set $\{A\in\mathcal{P}(X) \mid A_d \longrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d\in D}$ is included in the set $\{A\in\mathcal{P}(X) \mid \overline{\lim}_D A_d \subset \bar{A}$, and $\underline{\underline{\lim}}_D A_d \supset A^\circ\}$ and includes the set $\{A\in\mathcal{P}(X) \mid A = \overline{\lim}_D A_d = \underline{\underline{\lim}}_D A_d\}$. In general, the inclusions are proper. But in hyperspace $(\mathcal{H}, \mathbf{T}_{11}(\mathcal{H}))$, the set $\{H\in\mathcal{H} \mid H_d \longrightarrow H\}$ consisting of all limit points of net $\{H_d\}_{d\in D}$ in $\mathcal{H}$ equals the set $\{H\in\mathcal{H} \mid H=$

$\overline{\mathrm{lim}}_D H_d = \underline{\underline{\mathrm{lim}}}_D H_d\}$. It is obvious that $(\mathscr{H}, \mathbf{T}_{11}(\mathscr{H}))$ is a Hausdorff space.

(6) In hyperspace $(\mathscr{P}(X), \mathbf{T}_{12}(\mathscr{P}(X)))$, the set $\{A \in \mathscr{P}(X) \mid A_d \twoheadrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d \in D}$ in $\mathscr{P}(X)$ is included in the set $\{A \in \mathscr{P}(X) \mid \bar{A} = \overline{\mathrm{lim}}_D A_d = \underline{\mathrm{lim}}_D A_d\}$ and includes the set $\{A \in \mathscr{P}(X) \mid A = \overline{\mathrm{lim}}_D A_d = \underline{\mathrm{lim}}_D A_d\}$. In general, the inclusions are proper. But in hyperspace $(\mathscr{F}, \mathbf{T}_{12}(\mathscr{F}))$, the set $\{F \in \mathscr{F} \mid F_d \twoheadrightarrow F\}$ consisting of all limit points of net $\{F_d\}_{d \in D}$ equals the set $\{F \in \mathscr{F} \mid F = \overline{\mathrm{lim}}_D F_d = \underline{\mathrm{lim}}_D F_d\}$. It is obvious that $(\mathscr{F}, \mathbf{T}_{12}(\mathscr{F}))$ is a Hausdorff space.

(7) In hyperspace $(\mathscr{P}(X), \mathbf{T}_{21}(\mathscr{P}(X)))$, the set $\{A \in \mathscr{P}(X) \mid A_d \twoheadrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d \in D}$ is included in the set $\{A \in \mathscr{P}(X) \mid A^{\circ} = \overline{\overline{\mathrm{lim}}}_D A_d = \underline{\underline{\mathrm{lim}}}_D A_d\}$ and includes the set $\{A \in \mathscr{P}(X) \mid A\, \mathrm{lim}_D A_d = \mathrm{lim}_D A_d\}$. In general, the inclusions are proper. But in hyperspace $(\mathscr{G}, \mathbf{T}_{21}(\mathscr{G}))$, the set $\{G \in \mathscr{G} \mid G_d \twoheadrightarrow G\}$ consisting of all limit points of net $\{G_d\}_{d \in D}$ equals the set $\{G \in \mathscr{G} \mid G = \overline{\overline{\mathrm{lim}}}_D G_d = \underline{\underline{\mathrm{lim}}}_D G_d\}$. It is obvious that $(\mathscr{G}, \mathbf{T}_{21}(\mathscr{G}))$ is a Hausdorff space.

(8) In hyperspace $(\mathscr{P}(X), \mathbf{T}_{22}(\mathscr{P}(X)))$, the set $\{A \in \mathscr{P}(X) \mid A_d \twoheadrightarrow A\}$ consisting of all limit points of net $\{A_d\}_{d \in D}$ equals the set $\{A \in \mathscr{P}(X) \mid \underline{\mathrm{lim}}_D A_d \supset A \supset \overline{\overline{\mathrm{lim}}}_D A_d\} = (\mathrm{lim}_D A_d)^{\cdot} \cap (\mathrm{lim}_D A_d)^{\circ}$. We can obtain from this that, in general, $\mathbf{T}_{22}(\mathscr{P}(X))$ is not a Hausdorff space.

Remark: The results in (3), (4), (8) are always true for arbitrary topological space X.

COLLORY 2. Let X be a compact Hausdorff space. Then

(1) If a net $\{A_d\}_{d \in D}$ in $(\mathscr{P}(X), \mathbf{T}_{12}(\mathscr{P}(X)))$ is convergent, then it has unique closed (in X) limit point: $A = \overline{\mathrm{lim}}_D A_d = \underline{\mathrm{lim}}_D A_d$.

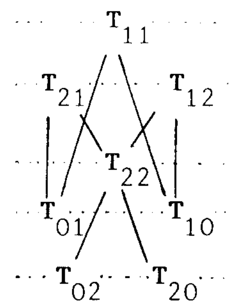
(2) If a net $\{A_d\}_{d \in D}$ in $(\mathscr{P}(X), \mathbf{T}_{21}(\mathscr{P}(X)))$ is convergent, then it has unique open (in X) limit point: $A = \overline{\overline{\mathrm{lim}}}_D A_d = \underline{\underline{\mathrm{lim}}}_D A_d$.

3. The Comparison between the Eight Kinds of Hypertopologies

THEOREM 3.1. Let X be a compact Hausdorff space. Then, in set $\mathscr{P}(X)$, the relation between the eight kinds of hypertopologies indicated in figue 1.

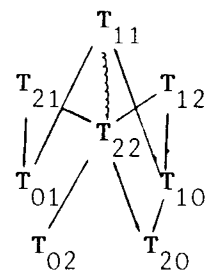
There the topologies upper are finer than the lower, and two topologies at the same level line (dotted line) are homeomorphic in following map:

$$c: \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$
$$A \longmapsto A^c \quad .$$



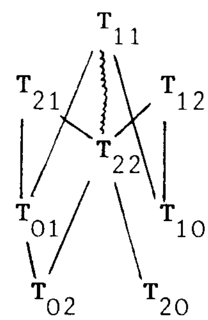
Figue 1. T_ω indicates $T_\omega(\mathcal{P}(X))$

THEOREM 3.2. Let X be a compact Hausdorff space. Then, in the collection $\mathcal{F}$ consisting of all closed subsets of X, the relation between the eight kinds of hypertopologies indicated in figue 2. There, the topologies upper are finer than the lower, and if $T_{11}(\mathcal{F})$ is compact, so is $T_{22}(\mathcal{F})$.



Figue 2. T_ω indicates $T_\omega(\mathcal{F})$

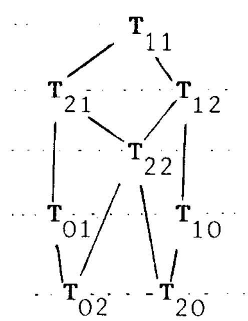
THEOREM 3.3. Let X be a compact Hausdorff space. Then, in the collection $\mathcal{G}$ consisting of all open subsets of X, the relation between the eight kinds of hypertopologies indicated in figue 3. There, the topologies upper are finer than the lower, and if $T_{11}(\mathcal{G})$ is compact, so is $T_{22}(\mathcal{G})$.



Figue 3. T_ω indicates $T_\omega(\mathcal{G})$

THEOREM 3.4. Let X be a compact Hausdorff space. Then, in the collection $\mathcal{H}$ consisting of all clopen subsets of X, the relation between the eight kinds of hypertopologies indicated in figue 4. There, the topologies upper are finer than the lower, and two topologies at the same level line (dotted line) are homeomorphic in the following map:

$$c: \mathcal{H} \longrightarrow \mathcal{H}$$
$$H \longmapsto H^c \quad .$$



Figue 4. T_ω indicates $T_\omega(\mathcal{H})$

Proot. If a net $\{H_d\}_{d \in D}$ converge to H in $(\mathcal{H}, T_{11}(\mathcal{H}))$, then $H = \overline{\lim}_D H_d = \underline{\underline{\lim}}_D H_d$, therefore, $\overline{\lim}_D H_d = \underline{\lim}_D H_d = H$, $\overline{\overline{\lim}}_D H_d = \underline{\underline{\lim}}_D H_d = H$, $\underline{\lim}_D H_d = \overline{\overline{\lim}}_D H_d = H$. Ttus we have $H_d \twoheadrightarrow H$, $H_d \twoheadrightarrow H$, and $H_d \twoheadrightarrow H$ in $\mathscr{H}$. So, $T_{11}(\mathscr{H})$ is finer than $T_{12}(\mathscr{H})$, $T_{21}(\mathscr{H})$ and $T_{22}(\mathscr{H})$. The rests of the proof is analogous, omitted.

4. Product Spaces of Hyperspaces

In this section, we will discuss the coincidence of the product topology of hypertopologies deduced from topological spaces X,Y and the hypertopology of product topology of them. If we discuss the problem by the means of neighbourhood, it will be difficult and complex. In this section, taking convergence as a tool, we can discuss the problem simply, and obtain some good results. In this section, if needed, we regard (A,B) as $A\times B\in\mathscr{P}(X\times Y)$, and let ${}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$ indicate the collection of all nonempty "rectangles" in $\mathscr{P}(X\times Y)$.

LEMMA 4.1. Let $\{A_d\times B_d\}_{d\in D}$ be a net in ${}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$. Then

$\underline{\lim}_D(A_d\times B_d)\supset A\times B\in{}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y) \iff \underline{\lim}_D A_d\supset A\in{}_0\mathscr{P}(X)$ and $\underline{\lim}_D B_d\supset B\in{}_0\mathscr{P}(Y)$.

Proof. Because $\underline{\lim}_D(A_d\times B_d)=\underline{\lim}_D A_d\times\underline{\lim}_D B_d$, the result is obvious.

LEMMA 4.2. Let $\{A_d\times B_d\}_{d\in D}$ be a net in ${}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$. Then

$\underline{\underline{\lim}}_D(A_d\times B_d)\supset A\times B\in{}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y) \iff \underline{\underline{\lim}}_D A_d\supset A\in{}_0\mathscr{P}(X)$ and $\underline{\underline{\lim}}_D B_d\supset B\in{}_0\mathscr{P}(Y)$.

LEMMA 4.3. Let $\{A_d\times B_d\}_{d\in D}$ be a net in ${}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$. Then

$\overline{\lim}_D A_d\subset A\in{}_0\mathscr{P}(X)$ and $\overline{\lim}_D B_d\subset B\in{}_0\mathscr{P}(Y) \implies \overline{\lim}_D(A_d\times B_d)\subset A\times B\in{}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$.

LEMMA 4.4. Let $\{A_d\times B_d\}_{d\in D}$ be a net in ${}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$. Then

$\overline{\overline{\lim}}_D A_d\subset A\in{}_0\mathscr{P}(X)$ and $\overline{\overline{\lim}}_D B_d\subset B\in{}_0\mathscr{P}(Y) \implies \overline{\overline{\lim}}_D(A_d\times B_d)\subset A\times B\in{}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$.

LEMMA 4.5. Let X be a compact Hausdorff space. Then for arbitrary net $\{A_d\}_{d\in D}\subset{}_0\mathscr{P}(X)$, we have $\overline{\lim}_D A_d\neq\phi$.

Proof. Because X is compact, so is $T_{12}({}_0\mathscr{P}(X))$[1],[2]. So there exists a subnet $\{A_{d'}\}_{d'\in D'}$ of net $\{A_d\}_{d\in D}$ such that $\{A_{d'}\}_{d'\in D'}$ is convergent in $({}_0\mathscr{P}(X),\ T_{12}(\mathscr{P}(X)))$. Thus, $\overline{\lim}_D A_d\supset\overline{\lim}_{D'}A_{d'}=\lim_{D'}A_{d'}\in{}_0\mathscr{P}(X)$, i.e., $\overline{\lim}_D A_d\neq\phi$.

LEMMA 4.6. Let X,Y be both compact Hausdorff spaces. $\{A_d\times B_d\}_{d\in D}$ be a net in ${}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y)$. Then $\overline{\lim}_D(A_d\times B_d)\subset A\times B\in{}_0\mathscr{P}(X)\times{}_0\mathscr{P}(Y) \implies \overline{\lim}_D A_d\subset A\in{}_0\mathscr{P}(X)$ and $\overline{\lim}_D B_d\subset B\in{}_0\mathscr{P}(Y)$.

Proof. Suppose $x\in\overline{\lim}_D A_d$ (because $\overline{\lim}_D A_d\neq\phi$), because $\overline{\lim}_D A_d=\bigcup\{\underline{\lim}_{D'}A_{d'}\ |$

$\{A_{d'}\}_{d'\in D'}$ is a subnet of $\{A_d\}_{d\in D}\}$, then $x\in \underline{\lim}_{D'} A_{d'}$ for some subnet $\{A_{d'}\}_{d'\in D'}$ of $\{A_d\}_{d\in D}$. Because Y is a compact Hausdorff space, $\overline{\lim}_{D'} B_{d'} \neq \phi$ (by lemma 4.5). Taking $y\in \overline{\lim}_{D'} B_{d'}$, then $y\in \underline{\lim}_{D''} A_{d''}$ for some subnet $\{A_{d''}\}_{d''\in D}$ Of net $\{A_{d'}\}_{d'\in D'}$. Thus we have $x\in \underline{\lim}_{D'} A_{d'} \subset \underline{\lim}_{D''} A_{d''}$, and $y\in \underline{\lim}_{D''} B_{d''}$. Therefore $(x,y)\in \underline{\lim}_{D''}(A_{d''}\times B_{d''}) \subset \overline{\lim}_{D}(A_d \times B_d) \subset A\times B$. So $x\in A$ and further $\overline{\lim}_D A_d \subset A \in {}_{\circ}\mathcal{P}(X)$. And same reason, we have $\overline{\lim}_D B_d \subset B \in {}_{\circ}\mathcal{P}(Y)$.

But in general, for a net $\{A_d \times B_d\}_{d\in D}$ in ${}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y)$, $\overline{\overline{\lim}}_D(A_d\times B_d) \subset A\times B \in {}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y) \not\Rightarrow \overline{\overline{\lim}}_D A_d \subset A \in {}_{\circ}\mathcal{P}(X)$ and $\overline{\overline{\lim}}_D B_d \subset B \in {}_{\circ}\mathcal{P}(Y)$.

THEOREM 4.1. Let X,Y be both compact Hausdorff spaces. Then

(1) $T_{10}({}_{\circ}\mathcal{F}(X))\times T_{10}({}_{\circ}\mathcal{F}(Y))$ is equivalent to $T_{10}({}_{\circ}\mathcal{F}(X)\times {}_{\circ}\mathcal{F}(Y))$.

(2) $T_{01}({}_{\circ}\mathcal{G}(X))\times T_{01}({}_{\circ}\mathcal{G}(Y))$ is equivalent to $T_{01}({}_{\circ}\mathcal{G}(X)\times {}_{\circ}\mathcal{G}(Y))$.

(3) $T_{02}({}_{\circ}\mathcal{P}(X))\times T_{02}({}_{\circ}\mathcal{P}(Y))$ is equivalent to $T_{02}({}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y))$.

(4) $T_{11}({}_{\circ}\mathcal{H}(X))\times T_{11}({}_{\circ}\mathcal{H}(Y))$ is equivalent to $T_{11}({}_{\circ}\mathcal{H}(X)\times {}_{\circ}\mathcal{H}(Y))$.

(5) $T_{12}({}_{\circ}\mathcal{F}(X))\times T_{12}({}_{\circ}\mathcal{F}(Y))$ is equivalent to $T_{12}({}_{\circ}\mathcal{F}(X)\times {}_{\circ}\mathcal{F}(Y))$.

Proof. According to the lemma 1 —— lemma 6 and the results in section 2, the proof is simple. So omitted.

LEMMA 4.7. Let $\{A_d\times B_d\}_{d\in D}$ be a net in ${}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y)$. Then $\phi \neq \overline{\overline{\lim}}_D(A_d\times B_d) = \underline{\underline{\lim}}_D(A_d\times B_d) = A\times B \in {}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y) \Longrightarrow \overline{\overline{\lim}}_D A_d = \underline{\underline{\lim}}_D A_d = A \in {}_{\circ}\mathcal{P}(X)$ and $\overline{\overline{\lim}}_D B_d = \underline{\underline{\lim}}_D B_d = B \in {}_{\circ}\mathcal{P}(Y)$.

The proof is elementary, so omitted.

THEOREM 4.2. Let X,Y be both compact Hausdorff spaces. Then $T_{21}({}_{\circ}\mathcal{G}(X))\times T_{21}({}_{\circ}\mathcal{G}(Y))$ is equivalent to $T_{21}({}_{\circ}\mathcal{G}(X)\times {}_{\circ}\mathcal{G}(Y))$.

Proof. According to the lemma 4.7, the proof is simple, omitted.

THEOREM 4.3. Let X,Y be both topological spaces. then $T_{20}({}_{\circ}\mathcal{P}(X))\times T_{20}({}_{\circ}\mathcal{P}(Y))$ is finer than $T_{20}({}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y))$. And $T_{22}({}_{\circ}\mathcal{P}(X))\times T_{22}({}_{\circ}\mathcal{P}(Y))$ is finer than $T_{22}({}_{\circ}\mathcal{P}(X)\times {}_{\circ}\mathcal{P}(Y))$.

Proof. According to the lemma 1 —— lemma 5, the proof is simple, omitted.

In general, $T_{22}({}_{\circ}\mathcal{G}(X))\times T_{22}({}_{\circ}\mathcal{G}(Y))$ properly finer than $T_{22}({}_{\circ}\mathcal{G}(X)\times {}_{\circ}\mathcal{G}(Y))$, even $T_{20}({}_{\circ}\mathcal{G}(X))\times T_{20}({}_{\circ}\mathcal{G}(Y))$ also properly finer than $T_{20}({}_{\circ}\mathcal{G}(X)\times {}_{\circ}\mathcal{G}(Y))$.

EXAMPLE: Let $X=Y=[0,1]$ equiped with usual topologies, to be both compact Hausdorff spaces. Q indicated the set consisting of all rational numbers in $[0,\frac{1}{2})$, and suppose $Q=\{r_1,r_2,\cdots\}$. Q_n indicates $\{r_1,r_2,\cdots r_n\}$ $(n=0,1,2,\cdots)$

Let $A_n=\begin{cases}[0,\frac{1}{2})\setminus Q_k & n=3k\\ [0,1] & n=3k+1\\ [0,\frac{1}{2}) & n=3k+2\end{cases}$ $\qquad B_n=\begin{cases}[0,1] & n=3k\\ [0,\frac{1}{2})\setminus Q_k & n=3k+1\\ [0,\frac{1}{2}) & n=3k+2\end{cases}$

Clearly, $A_n\in{}_0\mathcal{G}(X)$, $B_n\in{}_0\mathcal{G}(Y)$ $(n=0,1,2,\cdots)$ and $\overline{\overline{\lim}}_N(A_n\times B_n)=[0,\frac{1}{2})\times[0,\frac{1}{2})\subset[0,\frac{1}{2}]\times[0,\frac{1}{2}]=\underline{\lim}_N(A_n\times B_n)$. Thus $A_n\times B_n\twoheadrightarrow[0,\frac{1}{2})\times[0,\frac{1}{2})\in{}_0\mathcal{G}(X)\times{}_0\mathcal{G}(Y)$, but $\overline{\overline{\lim}}_N A_n=[0,1]\not\subset[0,\frac{1}{2}]=\underline{\lim}_N A_n$, and $\overline{\overline{\lim}}_N B_n=[0,1]\not\subset[0,\frac{1}{2}]=\underline{\lim}_N B_n$. So $\{A_n\}_{n\in N}$ and $\{B_n\}_{n\in N}$ are not convergent respectively in $\mathbf{T}_{22}({}_0\mathcal{G}(X))$ and $\mathbf{T}_{22}({}_0\mathcal{G}(Y))$. Therefore $\mathbf{T}_{22}({}_0\mathcal{G}(X))\times\mathbf{T}_{22}({}_0\mathcal{G}(Y))$ is is properly finer than $\mathbf{T}_{22}({}_0\mathcal{G}(X)\times{}_0\mathcal{G}(Y))$.

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