On fuzzy convergence classes*

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Abstracts

The concepts of the fuzzy point and its quasi-coincidence neighborhood have been introduced in [1]. Using these notions, we shall give the theory of fuzzy convergence classes. Especially the characterization of fuzzy topology via the fuzzy convergence classes and relative analysis are been made.

Keywords Fuzzy topology, Fuzzy convergence class, Q-neighborhood.

Introduction

After introducing the fundamental concepts of fuzzy point and its neighborhood structure (so called Q-neighborhood), we have attempt to get a characterization of fuzzy topology via the corresponding fuzzy convergence classes in [1]. The results obtained in [1] is basically parallel to that in general topology and is not perfect.

* This project is supported by the Science Fund of Chinese Academy of Science.

In fact, the range domain of set functions is now extended from the two elements set $\{0, 1\}$ to the unit interval $[0, 1]=I$, roughly speaking, the related problem is considered in one more dimensional space. Moreover the rang domain I does not be simply seen as a set of points, there is an order relation among the points of I. Therefore the desired characterization in fuzzy topology should be complicated somewhat than that in general topology. Considering properly the influence affected by the range domain I, we now get a new characterization. Several analyses about the new characterization will be also made.

Let X be ordinary non-empty set and the value domain be I. The concepts which do not define here we refer to [1].

1. Fuzzy convergence classes

Definition 1 Suppose that D is directed set, and for each $m \in D$ there are a directed set E^m and fuzzy net $S^m = S^m(n)$, $n \in E^m$. Then under product ordering we have a directed set $F = D \times \mathop{\times}\limits_{m \in D} E^m$ and a fuzzy net S defined by

$$S(m, f) = S^m(f(m)), \quad m \in D, \ f \in \mathop{\times}\limits_{m \in D} E^m.$$

The fuzzy net S is called induced net (exactly, associated with D and each S^m)

Definition 2 Let $\mathcal{G}$ be a class consisting of pairs (S, e), where $S = S(n)$, $n \in D$ is a fuzzy net in X and e is fuzzy point in X. $\mathcal{G}$ is called a convergence class for X iff it satisfies the five conditions lised below. For convenience, we say that S converges

($\mathcal{G}$) to e iff $(S, e) \in \mathcal{G}$.

(G1) The constant value net $\{S(n) \equiv$ some $e\}$ converges ($\mathcal{G}$) to e.

(G2) If $(S, e) \in \mathcal{G}$, then so does each fuzzy subnet of S.

(G3) If $(S, e) \notin \mathcal{G}$, then there is a fuzzy subnet T of S, no fuzzy subnet of which converges ($\mathcal{G}$) to e.

(G4) Let D be a directed set. For each $m \in D$, let the fuzzy net $S^m = \{S(m, n), n \in E^m\}$ converge ($\mathcal{G}$) to S(m) and let the fuzzy net $\{S(m), m \in D\}$ thus obtained, converge ($\mathcal{G}$) to e. Then the induced net (with respect to D and each S^m) also converges ($\mathcal{G}$) to e.

(G5) For each point $x \in X$ and real directed set $D \subseteq (0, 1]$, if $\rho \leq \sup D$, then the fuzzy net $\{x_\mu, \mu \in D\}$ converges ($\mathcal{G}$) to x_ρ.

Theorem 1 Let $(X, \mathcal{T})$ be a fuzzy topological space. Then the set of pairs $\{(S, e)$: fuzzy net S converges to $e\}$ is a fuzzy convergence class, denoted by $\varphi(\mathcal{T})$.

Proof In §14 of [1] we have already shown that $\varphi(\mathcal{T})$ satisfies the conditions above (G1)-(G4). As to (G5), write $\lambda = \sup D$, in view of that each Q-neighborhood of x_ρ is naturally Q-neighborgood of x_λ and that fuzzy net $\{x_\mu, \mu \in D\}$ clearly converges to x_λ, it follows that $\varphi(\mathcal{T})$ satisfies (G5).

Proposition 1 Let Ω be a family of fuzzy points in X and $A = \vee\Omega$. Let the set of pairs $\mathcal{G}$ satisfy the conditions (G4) and (G5). If fuzzy net S in A converges to e, then there exists a fuzzy net $\tilde{S}$ that consistes of fuzzy points in Ω and converges ($\mathcal{G}$) to e.

Proof Suppose that $S = \{S(m), m \in D\}$. For each $m \in D$, since

$\mathcal{S}(m)=y_\rho$ is in A, we have the family of fuzzy points $\{y_{\rho_n}\}\subseteq\Omega$ such that $y_\rho\leq\bigvee\{y_{\rho_n}\}$. Writting the set of these reals ρ_n as E^m, we get a fuzzy net $S^m=\{y_{\rho_n},\ \rho_n\in E^m\}$. By $\rho\leq\sup E^m$ and (G5), S^m converges $(\mathfrak{G})$ to $\mathcal{S}(m)$. Now from (G4) it is not difficult to give a induced net as desired net.

2. Characterization of fuzzy topology

Theorem 2 For each set of pairs $\mathfrak{G}$, a map $C:I^X\longrightarrow I^X$ is induced as follows: for $A\in I^X$, put $\mathfrak{G}(A)=\{e$: for certain net $\mathcal{S}$ in A, $(\mathcal{S},e)\in\mathfrak{G}\}$. Define $C(A)\equiv A^c=\bigvee\mathfrak{G}(A)$ ... (*) Now if $\mathfrak{G}$ is a fuzzy convergence class for X, then the following hold:

(1) The correspondence $A\longmapsto A^c$ is closure operator. The topology thus obained, is denoted by $\psi(\mathfrak{G})$.

(2) $\varphi(\psi(\mathfrak{G}))=\mathfrak{G}$

(3) $\psi\varphi\mathcal{T}=\mathcal{T}$, therefore there is a bijective map between the set of all fuzzy topologies for X and the set of all fuzzy convergence classes for X. Moreover, this map is order-reversing, i.e. if $\mathcal{T}_1\supseteq\mathcal{T}_2$, then $\varphi(\mathcal{T}_1)\subseteq\varphi(\mathcal{T}_2)$.

Proof. The argument is processed as follows:

(1) It is easy to see that $\emptyset^c=\emptyset$, $A\leq A^c$ and $(A\vee B)^c=A^c\vee B^c$. (Notice that for establishing $(A\vee B)^c\leq A^c\vee B^c$, we need to use the implication: $a\leq b\vee c\Rightarrow$ either $a\leq b$ or $a\leq c$. This fact is trivial for value domain I but is not true for more general value domain).

Now to show c is closure operator, it suffice to prove $(A^c)^c\leq A^c$. Take $e\in\mathfrak{G}(A^c)$, there is a fuzzy net $\mathcal{S}=\{\mathcal{S}(m),\ m\in D\}$ converging $(\mathfrak{G})$

to e. To show $e \leqslant A^c$, we shall get a fuzzy net $\tilde{S}$ in A converging $\mathcal{G}$ to e. Taking any $m \in D$, we denote the support point and the membership grade of $S(m)$ by x and λ respectively. Write $B_m = \{\rho$: $\exists$ fuzzy net in A converging ($\mathcal{G}$) to $x_\rho\}$. Obviously $\sup B_m = A^c(x) \geqslant \lambda$. By (G5), the fuzzy net $S^m = \{x_\rho, \rho \in B_m\}$ converges ($\mathcal{G}$) to $x_\lambda = S(m)$. By the definitio of B_m, for each $\rho \in B_m$, there exists fuzzy net in A converging ($\mathcal{G}$) to x_ρ. Thus in virture of (G4), we have a induced net T in A (with respect to B_m and others) converging ($\mathcal{G}$) to $S(m)$. That is to say, for each $m \in D$, there is a fuzzy net in A converging ($\mathcal{G}$) to $S(m)$. Again using (G4), we get a induced net $\tilde{S}$ in A (with spect to D) converging ($\mathcal{G}$) to e. This shows that c is closure operator.

(2) Suppose that $(S, e) \in \mathcal{G}$ and write $\mathcal{T} = \psi(\mathcal{G})$. We need to show that S converges to e relative to topology $\mathcal{T}$. Otherwise there is an open Q-neighborhood U such that the fuzzy net S is not coincident with U eventually, i.e. S is frequently contained in $U' = F$. Thus there is subnet T of S in F. In view of (G2), T converges ($\mathcal{G}$) to e. But F is closed subset, hence $e \leqslant F^c = F$. This contradictes with assumption that e coincides with $F' = U$.

(3) Suppose that fuzzy net S converging to e relative to topology $\mathcal{T}$. We need to show that S converges ($\mathcal{G}$) to e. Otherwise, by (G3), we have subnet $T = \{T(m), m \in E\}$ of S such that no subnet of T converges ($\mathcal{G}$) to e. Now by Theorem 1, T converges to e relative to $\mathcal{T}$. Put $A_m = \bigvee \{T(n) : n \geqslant m\}$ $(m \in E)$. By Theorem 13.3 of [1], $e \in A_m^c$. Write $e = x_\lambda$. For $m \in E$, put $D^m = \{\rho$: In A_m certain fuzzy net S^ρ con-

verging $(\mathcal{G})$ to x_ρ .} Furthmore, in view of Proposition 1, we can assume that for each $\rho \in D^m$ the fuzzy points of net S are the form $T(n)$ where $n \geqslant m$. Obviously $A_m^c(x)=\sup D^m \geqslant \lambda$. By (G5) the fuzzy net $\{x_\rho, \rho \in D^m\}$ converges $(\mathcal{G})$ to e. Thus we have induced net $\widetilde{T}^m$ (with respect to D^m and S) converging to e. The corresponding directed set of $\widetilde{T}^m$ is denoted by F^m. Clearly, the fuzzy points of $\widetilde{T}^m$ are the form $T(n)$ $(n \geqslant m)$. Again using (G4), we get a induced net $\widetilde{T}$ (with respect to E and $\widetilde{T}^m$) converging to e. The corresponding directed set of T is $E \times \bigtimes_{m \in E} F^m$. We claim that $\widetilde{T}$ is subnet of T. In fact, for any $m_1 \in E$, when $m \geqslant m_1$ we have $\widetilde{T}(m, f)=\widetilde{T}^m(f(m))$ for any $f \in \bigtimes_{n \in E} F^n$. Hence $T(m, f)$ is the form $T(n)$, where $n \geqslant m \geqslant m_1$. Thus we get a subnet $\widetilde{T}$ of T converging $(\mathcal{G})$ to e. This is a contradiction!

By (2) and (3) above, we have $\varphi\psi(\mathcal{G}) = \mathcal{G}$.

(4) To show $\psi\varphi(\mathcal{T})=\mathcal{T}$. For a fuzzy topology $\mathcal{T}$ for X, from Theorem 11.1 of [1], it follows that the closure operator induced by $\varphi(\mathcal{T})$ just is one associated with the topology $\mathcal{T}$. Namely, $\psi\varphi(\mathcal{T})=\mathcal{T}$.

(5) When $\mathcal{T}_1 \supseteq \mathcal{T}_2$, it is clear that $\varphi(\mathcal{T}_1) \subseteq \varphi(\mathcal{T}_2)$. We complete the proof of Theorem 2.

3. Some Analyses on (G5)

Comparing the conditions of the fuzzy convergence class with the ones in general topology, we find the condition (G5) is new.

We shall analyse the condition (G5) in details.

(I) The condition (G5) is necessary. In fact, we shall show by examples below that for set $\mathcal{G}$ of pairs satisfying just (G1)-(G4), the operator c defined by formula (*) in Theorem 2 may be not closure operator, i.e $(A^c)^c \neq A^c$; Moreover, for certain case the operator c is closure operator, but $\varphi\psi(\mathcal{G}) \neq \mathcal{G}$.

<u>Example 1.</u> Let X be singelton $\{x\}$. For each fuzzy net $S=\{S(n),\ n\in D\}$, writting the membership grade of $S(n)$ by λ_n we get a real net $\tilde{S}=\{\lambda_n, n\in D\}$ associated with S . For each $\lambda\in I$, the ordinary neighborhood system of real λ denotes $N(\lambda)$. Now we take a set of pairs $\mathcal{G}=\{(S, x_\lambda)\}$ as follows:

When $\lambda\neq 1,\ 2/3$, $\tilde{S}$ is eventually in U for $\forall\ U\in N(\lambda)$.

When $\lambda=2/3$, $\tilde{S}$ is eventually in $(U-\{1/3\})\cup\{2/3\}$ for $\forall\ U\in N(1/3)$.

When $\lambda=1$, $\tilde{S}$ is eventually in $(U-\{2/3\})\cup\{1\}$ for $\forall\ U\in N(2/3)$.

It easy to see that $\mathcal{G}$ satisfies the condition (G1)-(G4). Now taking $A=x_{1/3}$, for operator c defined by above formula (*), we have $A^c=x_{2/3}$, $(A^c)^c=x_1\neq A^c$.

<u>Example 2 .</u> Suppose that X is a non-empty set. $\mathcal{G}=\{(S, e)$: $S(n)$ is eventually equal to e $\}$. Obviously $\mathcal{G}$ satisfies the conditions (G1)-(G4). Meanwhile we have $A^c=A$ $(A\in I^X)$, i.e. the induced operator c is closure operator and the relative topology $\mathcal{T}$ is discrete topology. Consider the fuzzy net $S=\{x_{1-1/n+1}, n\in\mathbb{N}\}$ ($\mathbb{N}$ denotes the set of all positive interger, $x\in X$). Clearly S converges to x_1 relative topology $\mathcal{T}$ but does not converge ($\mathcal{G}$) to x_1.

(II) The condition (G5) can be decomposed into two parts as follows:

(G5)' For each $x \in X$ and each directed real set $D \subseteq (0, 1]$, the fuzzy net $\{x_\mu, \mu \in D\}$ converges $(\mathcal{G})$ to x_λ, where $\lambda = \sup D$.

(G5)" If $(S, x_\lambda) \in \mathcal{G}$, then for each $\mu \in (0, \lambda]$ we have $(S, x_\mu) \in \mathcal{G}$.

We shall show by examples that the condition (G5) can not be replaced by either (G5)' or (G5)". First, we see

Proposition 2 Suppose that $\mathcal{G}$ satisfies (G2), (G4) and (G5). If $(S, x_\lambda) \in \mathcal{G}$ and $\mu \in (0, \lambda]$ then $(S, x_\mu) \in \mathcal{G}$, i.e. the condition (G2)" holds.

Proof Suppose that $D = \{\lambda\}$ is singelton. By (G5), the fuzzy net $\{x_\lambda, \lambda \in D\}$ converges $(\mathcal{G})$ to x_μ. For $n \in E$, putting $\tilde{S}(\lambda, n) = S(n)$ and $E_\lambda = E$, we get fuzzy net $\tilde{S} = \{\tilde{S}(\lambda, n), n \in E_\lambda\}$. Obviously $\tilde{S}$ is a copy of S. By (G2), $\tilde{S}$ converges $(\mathcal{G})$ to x. Using (G4), we have a induced net T converging $(\mathcal{G})$ to x_μ (with respect to D and E_λ). Condiser the correspondence $n \mapsto (\lambda_n, f_n)$, where $n \in E$, and f_n sending λ to $n \in E_\lambda$. Clearly S is subnet of T. By (G2), S converges $(\mathcal{G})$ to x_μ.

Example 3 We adapt the example 1 above by adding following pairs (S, x_λ) to $\mathcal{G}$:

When $\lambda = 2/3$ or 1, $\tilde{S}$ is eventually in U for each $U \in N(\lambda)$.

This set of pairs obtained is denoted by $\tilde{\mathcal{G}}$. Obviously $\tilde{\mathcal{G}}$ satisfies (G1)-(G4) and (G5)'. For $A = x_{1/3}$, we have $A^c \neq (A^c)^c$. Note that the condition (G5)" does not hold, because for $S = \{x_{1-1/n+1}, n \in \mathbb{N}\}$ we have $(S, x_1) \in \tilde{\mathcal{G}}$ but $(S, x_{1/2}) \notin \tilde{\mathcal{G}}$.

Example 4 Suppose that $X=\{x\}$. The set of pairs $\mathcal{G}=\{(S, x_\lambda)\}$ where $S=\{x_{p_n}, n \in D\}$ satisfying the following conditions:

when $\lambda=1$, eventually $p_n=1$;

when $\lambda<1$, for $\forall \epsilon>0$, eventually $p_n>\lambda-\epsilon$. Obviously $\mathcal{G}$ satisfies (G1)-(G4) and (G5)". The operator c defined by the above formula (*) is identity mapping, hence the induced topology is discrete topology $\mathcal{T}$. But the fuzzy net $S=\{x_{1-1/n+1}, n \in N\}$ converges to x_1, relative the topology $\mathcal{T}$ and does not converge ($\mathcal{G}$) to x_1.

References

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