Cauchy problem under fuzzy control

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A control system, when the control is unknown and one may only guess about its possible values, is considered. The formalization is based on a fuzzy function theory.

Let n be a positive integer. Denote by R^n the n-vector space with Euclidean inner product $\langle \cdot , \cdot \rangle$, Euclidean norm $|\cdot|$ and the Lebesgue measure. Let $x, y, x_H \in R^n$ and $\gamma \geqslant 0$, $\tau_1 > 0$. Let $i = 0, 1, 2, \ldots$ and $\lambda, \lambda_0 \in [0,1]$, $\lambda \neq 0$.

Denote by T the segment $[0, \tau_1]$ of R^1 and by S^* the closed unit ball (about the origin) of R^n. Let $\tau, \theta \in T$ and $x^*, x_0^* \in S^*$, $x^* \neq 0$.

Let

$$(\tau, x, y) \to g_\tau(x, y) : T \times R^{2n} \to R^n$$

be a function continuous in (τ, x, y) and Lipschitzian in (x, y) with a constant Γ . Let A be an orthogonal $n \times n$-matrix and A^* be the transposed one, with E being the identity matrix.

1. Cauchy problem under vector control

A vector (-valued) function is defined as a mapping of T to R^n. Let $x_.$ and $y_.$ be vector functions. Let

$$\|x_. - y_.\|_\gamma = \sup_\tau e^{-\gamma\tau} |x_\tau - y_\tau| .$$

Let $u_.$ and $v_.$ be continuous vector functions.

Consider the Cauchy problem under vector control

$$(g, u, x_н) \quad \begin{cases} \frac{d}{d\tau} x = g_\tau(x, u), \quad u = u_\tau , \\ x_0 = x_н . \end{cases}$$

A vector function $x_.$ is called a solution of the problem $(g, u, x_н)$ if it is differentiable and such that

$$\frac{d}{d\tau} x_\tau = g_\tau(x_\tau, u_\tau) \quad \forall \tau , \qquad x_0 = x_н .$$

Theorem. A solution of $(g, u, x_н)$ exists and is unique. If $x_.$ and $y_.$ are the solutions of $(g, u, x_н)$ and $(g, v, x_н)$ respectively, then

$$\|x_. - y_.\|_\Gamma \leq \Gamma \int_0^{\tau_1} |u_\tau - v_\tau| d\tau .$$

The first part of the theorem is known [5] to follow from Banach's contraction principle. The second one is a consequence of Gronwall's inequality.

2. Space of fuzzy sets

Consider a function from R^n to $[0,1]$ that takes the value 1 at least once, has the bounded support, is quasi-concave and upper semi-continuous. Denote by $\mathcal{L}\mathcal{O}$ the set of all such functions. Let $\mu(\cdot), \nu(\cdot) \in \mathcal{L}\mathcal{O}$.

Given $\mu(\cdot)$, consider the function

$$x_0^* \to \mu^*(x_0^*) = \max_x \{\langle x, x_0^* \rangle : \mu(x) \geq |x_0^*|\}.$$

Denote by $\mathcal{L}\mathcal{O}^*$ the set of all such functions.

Lemma. A function $\eta^*(\cdot) : S^* \to R^1$ belongs to $\mathcal{L}\mathcal{O}^*$ if and only if it is

1) equal to 0 at the origin;

2) 'semi-additive', i. e. the function

$$(\lambda, x) \to H_\lambda^*(x) = \begin{cases} 0, & x = 0, \\ \dfrac{|x|}{\lambda} \eta^*\left(\lambda \dfrac{x}{|x|}\right), & x \neq 0, \end{cases}$$

is semi-additive in x;

3) 'semi-homogenious', i. e.

$$\lambda \eta^*(x^*) \leq \eta^*(\lambda x^*) \quad \forall\ (\lambda, x^*);$$

4) bounded in the functional semi-norm $\|\cdot\|$, i. e.

$$\|\eta^*(\cdot)\| = \sup_{x^*} \frac{1}{|x^*|} |\eta^*(x^*)| < \infty;$$

5) upper semi-continuous.

The duality $\mu^*(\cdot) \to \mu(\cdot)$ is implemented by the formula

$$\mu(x) = \max\{\lambda_0 : \langle x, x_0^* \rangle \leqslant \mu^*(x_0^*) \ \forall\, |x_0^*| = \lambda_0\}.$$

See the 'non-fuzzy' case [7§13] as crucial for the proof.

The pair $(\mu(\cdot), \mu^*(\cdot))$ will be referred to as a non-empty bounded convex closed fuzzy subset μ of R^n (or, briefly, a fuzzy set μ) with the characteristic function $\mu(\cdot)$ and support function $\mu^*(\cdot)$. Denote by $\mathfrak{M}$ the set of all fuzzy sets.

The fuzzy set δ_x with the support function

$$x_0^* \to \langle x, x_0^* \rangle$$

will be referred to as the set concentrated at x.

Equip the set $\mathfrak{M}$ with an equality $=$, an addition $+$, a multiplication by γ and a metric ρ by the formulae

$$\mu = \nu \iff \mu^*(\cdot) = \nu^*(\cdot),$$

$$(\mu + \nu)^*(\cdot) = \mu^*(\cdot) + \nu^*(\cdot),$$

$$(\gamma\mu)^*(\cdot) = \gamma\mu^*(\cdot),$$

$$\rho(\mu, \nu) = \sup_{x^*} \frac{1}{|x^*|} \left| \mu^*(x^*) - \nu^*(x^*) \right|.$$

Appropriate attributes are obvious. See also [2].

3. Fuzzy functions

A fuzzy function will be defined as a mapping of $\mathcal{T}$ to $\mathfrak{M}$. Let $\mu_.$ and $\nu_.$ be fuzzy functions. Whenever $\mu_\tau \equiv \nu_\tau$ let us write $\mu_. = \nu_.$. Let

$$\|\mu_. - \nu_.\|_\gamma = \sup_\tau e^{-\gamma\tau} \rho(\mu_\tau, \nu_\tau) .$$

A fuzzy function will be called continuous if it is continuous as a mapping of $\mathcal{T}$ to $(\mathfrak{M}, \rho)$. Let $\zeta_.$ and $\eta_.$ be continuous fuzzy functions.

A fuzzy function $\mu_.$ will be called differentiable (integrable) if there exists a fuzzy function $d\mu_./d\tau$ (resp. $\int_0^{\cdot} \mu_\theta \, d\theta$) such that

$$\left(\frac{d}{d\tau}\mu_\tau\right)^*(x^*) = \frac{\partial}{\partial\tau}\mu_\tau^*(x^*)$$

$$\left(\left(\int_0^\tau \mu_\theta \, d\theta\right)^*(x^*) = \int_0^\tau \mu_\theta^*(x^*)\, d\theta\right) \quad \forall\, (\tau, x^*).$$

The fuzzy function $\tau \to \delta_{x_\tau}$ will be called the function concentrated at $x_.$.

Appropriate attributes are obvious.

4. Convex fuzzification

Define the correspondence

$$(\mu, \nu) \to \operatorname{co} g_\tau(\mu, \nu) : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$$

by the formula

$$(\mathrm{co}\, g_\tau(\mu,\nu))^*(x_0^*)$$

$$= \max_{x,y} \{ \langle g_\tau(x,y), x_0^* \rangle : \min\{\mu(x),\nu(y)\} \geqslant |x_0^*| \}.$$

It will be called the convex fuzzification of the function $g_\tau(\cdot,\cdot)$.

Define additionally the product of μ by A as a fuzzy set $A\mu$ equal to $\mathrm{co}\, A\mu$.

Remark. It is evident that

$$\mu + \nu = \mathrm{co}(\mu + \nu), \quad \gamma\mu = \mathrm{co}(\gamma\mu).$$

Let us make the agreement to identify the pair (μ,ν) and the non-empty bounded convex closed fuzzy subset of R^{2n} with the characteristic function

$$(x,y) \to \min\{\mu(x),\nu(y)\}$$

(like one identifies pairs (x,y) and elements of R^{2n}).

Lemma. The mapping

$$\mathrm{co}\, g_\cdot(\cdot,\cdot) : T \times (\mathcal{M}\times\mathcal{M}, \rho) \to (\mathcal{M}, \rho)$$

is continuous in (τ,μ,ν) and Lipschitzian in (μ,ν) with the (same) constant Γ.

The proof appeals to the commonplace 'non-fuzzy' case.

5. Cauchy problem under fuzzy control

Consider now the Cauchy problem under fuzzy control

$$(g, \jmath, x_H) \quad \begin{cases} \frac{d}{d\tau} x = g_\tau(x, u), \quad u \in \jmath_\tau, \\ x_0 = x_H . \end{cases}$$

A fuzzy function $\mu.$ will be called a solution of the problem $(g, \jmath, x_H)$ if it is differentiable and such that

$$\frac{d}{d\tau}\mu_\tau = \operatorname{co} g_\tau(\mu_\tau, \jmath_\tau) \quad \forall \tau , \quad \mu_0 = \delta_{x_H} .$$

Theorem. A solution of $(g, \jmath, x_H)$ exists and is unique (up to the $=$). If $\mu.$ and $\nu.$ are the solutions of $(g, \jmath, x_H)$ and (g, η, x_H) respectively, then

$$\| \mu. - \nu. \|_\Gamma \leq \Gamma \int_0^{\tau_1} \rho(\jmath_\tau, \eta_\tau)\, d\tau .$$

Indeed, the relevant mathematical structures are mutually coordinated to the extent to apply the standard scheme based on Banach's principle and Gronwall's inequality.

Theorem. If $\mu.$ and $x.$ are the solutions of $(g, \jmath, x_H)$ and (g, u, x_H) respectively, then

$$\jmath. = \delta_{u.} \Rightarrow \mu. = \delta_{x.} .$$

The proof is straightforward.

6. Linear problem

As a particular Cauchy problem under fuzzy control let us consider the problem

$$(A, \mathfrak{J}, x_H) \quad \begin{cases} \dfrac{d}{d\tau} x = Ax + u , & u \in \mathfrak{J}_\tau , \\ x_0 = x_H . \end{cases}$$

Theorem. The solution of $(A, \mathfrak{J}, x_H)$ has the form

$$\tau \to \delta_{e^{\tau A} x_H} + \int_0^\tau \sum_i \frac{(\tau-\theta)^i}{i!} (A^i \mathfrak{J}_\theta)\, d\theta ,$$

taking into account that the formal functional series

$$\theta \to \sum_i \frac{(\tau-\theta)^i}{i!} (A^i \mathfrak{J}_\theta) \in (\mathfrak{M}, +, \rho)$$

is a continuous - in order to be integrable - fuzzy function (since the series consists of continuous fuzzy functions and converges uniformly). If $\mu_.$ and $\nu_.$ are the solutions of $(A, \mathfrak{J}, x_H)$ and (A, η, x_H) respectively, then

$$\| \mu_. - \nu_. \|_1 \leq \int_0^{\tau_1} \rho(\mathfrak{J}_\tau, \eta_\tau)\, d\tau .$$

The proof is straightforward. The solution form is determined by the method of successive approximations.

For example, the solution of $(E, \mathfrak{J}, 0)$ has the form $\tau \to (e^\tau - 1)\mathfrak{J}_0$ as soon as $\mathfrak{J}_\tau \equiv \mathfrak{J}_0$.

7. Interpretation

Given solutions $\mu_.$ and $x_.$ of $(g, \jmath, x_H)$ and (g, u, x_H) respectively, consider a functional $x_. \to \iota(x_.) \in [0,1]$ such that, for any $x_.$, the following holds:

$$\mu_\tau(x_\tau) = 1 \quad \forall \tau \quad \Longrightarrow \quad \iota(x_.) = 1 .$$

Two examples:

$$\iota(x_.) = \inf \mu_\tau(x_\tau),$$

$$\iota(x_.) = \frac{1}{\tau_1} \int_0^{\tau_1} \mu_\tau(x_\tau)\, d\tau \qquad \forall x_.$$

(the integral exists because the scalar function $\tau \to \mu_\tau(x_\tau)$ turns out to be upper semi-continuous).

The number $\iota(x_.)$ should be regarded as a subjective evaluation of the extent to which the vector function $x_.$ can be a solution of the Cauchy problem under secret preset control

$$(g, ?, x_H) \qquad \begin{cases} \dfrac{d}{d\tau} x = g_\tau(x, u), \quad u = ?, \\[2ex] x_0 = x_H . \end{cases}$$

'Subjectivity' consists, at least, in choice of concrete $\jmath$. and $\iota(\cdot)$.

Given problems $(E, \jmath, 0)$ and $(E, u, 0)$, in both the examples we have $\iota(x_.) = \jmath_0(u_0)$ as soon as $\jmath_\tau \equiv \jmath_0$ and $u_\tau \equiv u_0$.

Index of related notions

References

1. Artstein Z. On the calculus of closed set-valued functions. - Indiana Univ. Math. J., 1974, 24, n° 5, 433--441
2. Bobylev V. N. Fuzzy-set spaces with Hausdorff's metric... - BUSEFAL, 1984, n° 18, 56--61
3. Bradly M., Datko R. Some analytic and measure theoretic properties of set-valued functions. - SIAM J. Contr. and Optim., 1977, 15, n° 4, 625--635
4. Brandão Lopes Pinto A. J., De Blasi F. S., Iervolino F. Uniqueness and existence theorems for differential equations with compact convex valued solutions. - Boll. Un. Mat. Ital., 1970, 3, n° 1, 47--54
5. Dugundji J., Granas A. Fixed point theory, 1. - Polish Scientific Publishers, 1982, p. 26
6. Matheron G. Random sets and integral geometry. - Wiley, 1975, 261 p.
7. Rockafellar T. Convex analysis. - Princeton Univ. Press, 1970, 451 p.