ON ANY CLASS OF FUZZY PREFERENCE RELATIONS IN REAL LINE II

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4. The main assumptions of SFP in real line

4.1. Consistence of SFP

Let $\hat{R}$ be set of all real numbers with the infinity and the minus infinity. If unfuzzy PR in $\hat{R}$ is the relation $\leqslant$ then we can represent the following table of interpretations:

	PR	PR_e	PR_s
R	$\leqslant$	$=$	$<$
$\bar{R}$	$>$	$\neq$	$\geqslant$
R^{-1}	$\geqslant$	$=$	$>$
$\overline{R^{-1}}$	$<$	$\neq$	$\leqslant$

We see that the PR $\leqslant$ satisfies the next conditions:

$PR_e = PR_e^{-1}$, $PR_e = \overline{PR_e^{-1}}$, $PR = \overline{PR_s^{-1}}$, $\overline{PR} = PR_s^{-1}$, $PR^{-1} = \overline{PR_s}$, $\overline{PR^{-1}} = PR_s$.

Let us displace this notice to the domain of FPR in $\hat{R}$. Let SFP in $\hat{R}$ satisfies the next conditions:

$$\varrho_e = \varrho_e^{-1} \quad (4.1), \qquad \bar{\varrho}_e = \overline{\varrho_e^{-1}} \quad (4.2),$$

$$\varrho = \overline{\varrho_s^{-1}} \quad (4.3), \qquad \bar{\varrho} = \varrho_s^{-1} \quad (4.4),$$

$$\varrho^{-1} = \bar{\varrho}_s \quad (4.5), \qquad \overline{\varrho^{-1}} = \varrho_s \quad (4.6).$$

The conditions (4.1) and (4.2) are always true, because they follow immediately from the symmetry of FPR_e.

Further, it is very easy verify that the conditions (4.3)-(4.6) are equivalent and they are equivalent to the next condition

$$\varrho \geqslant \overline{\varrho^{-1}} \tag{4.7}$$

Therefore, we propose to mark off the following class of FPR.

Definition 4.1: Any FPR in $\hat{R}$ satysfing (4.7) is called consistent FPR.

Any FSP in $\hat{R}$ generated by consistent FPR we shall call consistent SFP. We have for it.

Lemma 4.1: Any consistent FPR is reflexive.

Proof: By consistency of FPR we get

$$\delta[\varrho] \geqslant \delta[\overline{\varrho^{-1}}] = 1 - \delta[\varrho].$$

So, the fuzzy subset $\delta[\varrho]$ is a W-universum in $\hat{R}$. It proves that FPR ϱ is reflexive. ■

Lemma 4.2: Any antisymmetrical FPR_s in $\hat{R}$ generates a consistent SFP such that FPR and FPR_e are defined by (4.3) and

$$\varrho_e = \overline{\varrho}_s \wedge \overline{\varrho_s^{-1}} \tag{4.8}$$

Proof: By (4.3) and (4.6) we obtain

$$\varrho = \overline{\varrho_s^{-1}} = 1 - \varrho_s^{-1} \geqslant \varrho_s = \overline{\varrho^{-1}}$$

for every antisymmetrical FPR_s. The identity (4.8) follows from (3.1), (4.3) and (4.5). ■

4.2. Monotonicity of SFP

The unfuzzy PR $\leqslant$ is described by membership function $\varrho : \hat{R}^2 \to \{0,1\}$, too. Often, the PR is interpreted as a family of closed interval in $\hat{R}$. Then the mappings $\varrho(\cdot,x) : \hat{R} \to \{0,1\}$ and $\varrho(y,\cdot) : \hat{R} \to \{0,1\}$ describe respectively the unfuzzy intervals $[-\infty, x]$ and $[y, +\infty]$. Analogically, the mappings $\varrho_s(\cdot,x) : \hat{R} \to \{0,1\}$ and $\varrho_s(y,\cdot)$ describe the unfuzzy one-side open intervals $[-\infty, x[$ and $]y, +\infty]$. As we know, if $x<y$ then $[-\infty,x] \subset [-\infty,y[$ and $[y,+\infty] \subset]x,+\infty]$.

Displacing this notice to the domain of SFP we mark off the following class of one.

Definition 4.2: If the SFP $(\varrho, \varrho_e, \varrho_s)$ fulfils the next conditions

$$\varrho(\cdot,x) \leqslant \varrho_s(\cdot,y) \quad , \tag{4.9}$$

$$\varrho(y,\cdot) \leqslant \varrho_s(x,\cdot) \quad , \tag{4.10}$$

for every real number x,y which $x<y$ then the SFP is called a monotonical one.

Additionally we propose to accept the following definition.

Definition 4.3: Let $\psi : \hat{R}^2 \to [0,1]$ be the membership function of FR. An FR satisfying for every x,y which $x<y$ conditions

$$\psi(\cdot,x) \leqslant \psi(\cdot,y) \ , \tag{4.11}$$

$$\psi(x,\cdot) \geqslant \psi(y,\cdot) \tag{4.12}$$

is called a monotonic FR.

For any monotonic SFP we have

Lemma 4.3: If the SFP is monotonic then the FPR is monotonic.

Proof: From (4.9) we obtain

$$\varrho(\cdot,x) \leqslant \varrho_s(\cdot,y) = \varrho(\cdot,y) \wedge (1 - \varrho(y,\cdot)) \leqslant \varrho(\cdot,y)$$

for each pair x,y such that $x<y$. The condition (4.12) we obtain by analogous way from (4.10). ■

Lemma 4.4: If the SFP $(\varrho, \varrho_e, \varrho_s)$ is monotonic, then we have

$$\varrho_s(y,x) \leqslant \varrho(y,x) \leqslant \tfrac{1}{2} \tag{4.13}$$

for every real numbers x,y which $x<y$.

Proof: By monotonicity of the SFP we get

$$\varrho_s(y,x) \leqslant \varrho(y,x) \leqslant \varrho_s(y,y) = \varrho(y,y) \wedge (1 - \varrho(y,y)) \leqslant \tfrac{1}{2}$$

for every x,y such that $x<y$. ■

4.3. The main definitions

If we take into account above assumptions then we have

Lemma 4.5: If SFP is consistent then the next conditions

$$\varrho(\cdot,x) + \varrho(y,\cdot) \leqslant 1 \tag{4.14}$$

$$\varrho_s(x,\cdot) + \varrho_s(\cdot,y) \geqslant 1 \tag{4.15}$$

for every real numbers x,y which $x<y$ and the conditions (4.9) and (4.10) are equivalent.

All above equiponderances are self-evident. On the face of all foregoing considerations we propose the following definitions.

Definition 4.4. Any consistent FPR in $\hat{R}$ satisfying the condition 4.14 is called a fuzzy relation "less or equal" (FLE).

Definition 4.5: The FPR_e generated by FLE is called a fuzzy relation "equal" (FEQ).

Definition 4.6: The FPR_s generated by FLE is called a fuzzy relation "less than" (FLT).

Definition 4.7: The SFP generated by FLE is called a system of fuzzy arrangement (SFA).

This notions fulfil the next thesis.

Theorem 4.1: Any FLE is well-defined FPR.

Proof: The reflexivity of FLE is showed in the Lemma 4.1. For any $(x,y,z) \in \hat{R}^3$ we get:

- if $x \leqslant z$ then

$$(1 - \varrho(x,z) \wedge \varrho(z,y)) \vee \varrho(x,y) \geqslant (1 - \varrho(z,y)) \vee \varrho(x,y) \geqslant$$
$$\geqslant (1 - \varrho(x,y)) \vee \varrho(x,y) \geqslant \frac{1}{2},$$

- if $y \geqslant z$ then

$$(1 - \varrho(x,z) \wedge \varrho(z,y)) \vee \varrho(x,y) \geqslant (1 - \varrho(x,z)) \vee \varrho(x,y) \geqslant$$
$$\geqslant (1 - \varrho(x,y)) \vee \varrho(x,y) \geqslant \frac{1}{2},$$

- if $y < z < x$ then

$$(1 - \varrho(x,z) \wedge \varrho(z,y)) \vee \varrho(x,y) \geqslant \frac{1}{2} \vee \varrho(x,y) \geqslant \frac{1}{2}.$$

So, the fuzzy subset $(1 - \varrho(\cdot,z) \wedge \varrho(z,\cdot)) \vee \varrho(\cdot,\cdot)$ is a W-universum in $\hat{R}^2$. It proves that the FLE is a fuzzy quasi-order relation. ■

Theorem 4.2: If antisymmetrical FR in $\hat{R}$ satisfies the condition (4.15) then it generates the SFA (FLE,FEQ,FLT) such that the FR is FLT.

Proof: The thesis we get by the Lemmas 4.2 and 4.5. ■

Theorem 4.3: FLE and FLT are monotonic FR. Moreover, their membership functions satisfy the next inequalities

$$\varrho(x,y) \geqslant \varrho_s(x,y) \geqslant \varrho(y,y) \geqslant \frac{1}{2} \geqslant \varrho_s(y,y) \geqslant \varrho(y,x) \geqslant \varrho_s(y,x) \quad (4.16)$$

for every pair $(x,y) \in \hat{R}^2$ which $x<y$.

Proof: The first thesis follows from the Lemma 4.3 and from the identity (4.6). The consistency of SFA implies

$$\varrho(y,y) \geqslant \frac{1}{2} \quad . \qquad (*)$$

Therefore, by monotonicity of SFA and by the identity (3.2) we obtain

$$\varrho(x,y) \geqslant \varrho_s(x,y) \geqslant \varrho(y,y) \geqslant \frac{1}{2}$$

for every $(x,y) \in \hat{R}^2$ which $x<y$. The inequalities (*) along with the identity (4.6) implies that

$$\frac{1}{2} \geqslant \varrho_s(y,y) \quad .$$

This **together** with the monotonicity of SFA and the identity (4.6) puts on end to proof of second thesis. ■

5. Suplementary assumptions of SFA

5.1. Quasi-antisymmetry of FLE

The crisp relation "less or equal" satisfies the axioms describing FLE. The crisp relation "less than" fulfils this axioms, too. We see that the Definition 4.4. is more general. As we known, the crisp relation "less or equal" is quasi-antisymmetrical. The crisp relation "less than" has not this property. Therefore, we propose to mark off the next class of FLE.

Definition 5.1: Each quasi-antisymmetrical FLE is called a strict FLE.

The SFA generated by strict FLE we shall call a strict SFA. We have the following thesis on strict FLE.

Theorem 5.1: Any FLE is strict iff it fulfils the next condition

$$\varrho(y,x) < \frac{1}{2} \tag{5.1}$$

for every pair $(x,y) \in \hat{R}^2$ which $x < y$.

Proof: The definition of quasi-antisymmetry for FLE can be expressed equivalently as follows:

$$\varrho(x,y) \wedge \varrho(y,x) \geqslant \frac{1}{2} \Rightarrow x = y$$

for all pairs $(x,y) \in \hat{R}^2$. This is equivalent to

$$x \neq y \Rightarrow \varrho(x,y) \wedge \varrho(y,x) < \frac{1}{2}. \qquad (*)$$

If $x < y$ then by means of monotonicity of FLE we get

$$\frac{1}{2} > \varrho(x,y) \wedge \varrho(y,x) \geqslant \varrho(y,x) \wedge \varrho(y,x).$$

On the other side, the condition (5.1) implies (*). The proof is complete. ■

Theorem 5.2: Any FLE is strict iff generated by it FLT satisfies the following condition

$$\varrho_s(x,y) > \frac{1}{2} \tag{5.2}$$

for all pairs $(x,y) \in \hat{R}^2$ which $x < y$.

Proof: The thesis follows immediately from the consistency of SFA and from (5.1). ■

Therefore, each FLT satisfying (5.2) we shall call a strict FLT. Obviously, each strict FLT generates by the identity (4.3) a strict FLE.

5.2. Continuity from above of FLE

Let $\{x_n\} \downarrow x$ and $\{y_n\} \uparrow y$. Then we observe that $\{[-\infty, x_n]\} \downarrow [-\infty, x]$ and $\{[y_n, +\infty]\} \downarrow [y, +\infty]$. Displacing this notice to the domain of FLE we mark off the following class of one.

Definition 5.2: Any FLE fulfilling the next conditions

$$\{\varrho(\cdot, x_n)\} \downarrow \varrho(\cdot, x) \quad , \tag{5.3}$$

$$\{\varrho(y_n, \cdot)\} \downarrow \varrho(y, \cdot) \tag{5.4}$$

is called a continuous from above FLE.

Theorem 5.3: Any FLE is continuous from above iff generated by it FLT is continuous from below i.e.

$$\{\varrho_s(x_n, \cdot)\} \uparrow \varrho_s(x, \cdot) \quad , \tag{5.5}$$

$$\{\varrho_s(\cdot, y_n)\} \uparrow \varrho_s(\cdot, y) \quad . \tag{5.6}$$

Proof: This thesis follows from consistence of SFA. ■

Theorem 5.4: If FLE is continuous from above then we have

$$\{\varrho_s(\cdot, x_n)\} \downarrow \varrho(\cdot, x) \ , \tag{5.7}$$

$$\{\varrho_s(y_n, \cdot)\} \downarrow \varrho(y, \cdot) \ , \tag{5.8}$$

$$\{\varrho(x_n, \cdot)\} \uparrow \varrho_s(x, \cdot) \ , \tag{5.9}$$

$$\{\varrho(\cdot, y_n)\} \uparrow \varrho_s(\cdot, y) \ . \tag{5.10}$$

for such sequences $\{x_n\}$ and $\{y_n\}$ that $x_n > x$ and $y_n < y$ for every positive integer n .

Proof: From monotonicity of SFA we obtain

$$\varrho(\cdot, x) \leqslant \varrho_s(\cdot, x_n) \leqslant \varrho(\cdot, x_n) \ .$$

It proves (5.7). Similarly we can prove the next thesis. ■

5.3. Real line unfuzzily bounded

Finally, let us consider the following class of FLE.

Definition 5.3: Let $\hat{R}^{UNSD}$ and $\hat{R}^{UNSU}$ are unfuzzy nonstrict dominant and unfuzzy nonstrict undominant in $\hat{R}$ generated by FLE. If $\hat{R}^{UNSD}$ and $\hat{R}^{UNSU}$ are not empty sets, then the FLE is called an unfuzzily bounding FLE.

It is very easy check that the FLE unfuzzily bounds the real line $\hat{R}$, because we have:

Theorem 5.5: The next conditions are equivalent:

a/ $\hat{R}^{UNSD} \neq \emptyset$

b/ $\exists_{y \in \hat{R}} \quad \varsigma(+\infty, y) = 1$

c/ $\varsigma(+\infty, +\infty) = 1$

d/ $\forall_{x \in \hat{R}} \quad \varsigma(x, +\infty) = 1$

e/ $\forall_{x \in \hat{R}} \quad \varsigma_s(+\infty, x) = 0$

f/ $\varsigma_s(+\infty, +\infty) = 0$

Proof: From monotonicity of FLE we obtain $\mu^{NSD}(\cdot) = \varsigma(+\infty, \cdot)$. This result along with definition of unfuzzy nonstrict dominant proves that, the conditions (a) and (b) are equivalent. Since the FLE is monotonic, the conditions (b), (c) and (d) are equivalent, too. From consistency of FLE we get the equiponderance for the conditions (d) and (e). The last equiponderance follows from monotonicity of FLT. ■

Theorem 5.6: The next conditions are equivalent:

a/ $\hat{R}^{UNSU} \neq \emptyset$

b/ $\exists_{y \in \hat{R}} \quad \varsigma(y, -\infty) = 1$

c/ $\varsigma(-\infty, -\infty) = 1$

d/ $\forall_{x \in \hat{R}} \quad \varsigma(-\infty, x) = 1$

e/ $\forall_{x \in \hat{R}} \quad \varsigma_s(x, -\infty) = 0$

f/ $\varsigma_s(-\infty, -\infty) = 0$

The proof of above theorem is similar to the last one.

Remark: All presented above assumptions and results will be employed in domain of fuzzy intervals and further in the theory of probability on fuzzy real line.

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