GENERALIZED SOLVABILITY CRITERIA FOR SYSTEMS OF FUZZY EQUATIONS DEFINED WITH t - NORMS

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For the solvability criterion of Sanchez (1984) for some types of equations in the field of fuzzy arithmetic a generalization was sketched in the previous paper Gottwald (1983). Here further extensions will be sketched:

- we consider also another type of fuzzy equations which is related to fuzzy controllers;
- we consider any t-norm instead of the minimum as in Sanchez (1984) and of minimum and algebraic product as in Gottwald (1983);
- we extend the results to systems of those equations too and hence also generalize the result of Gottwald (1984).

For logical and set theoretical notation which is not explained here the reader is referred to Gottwald (1983). Additionally, we suppose that **t** is any t-norm which acts as the truth function of a conjunction connective $\&_t$ in our (many valued) language. With each such t-norm **t** a so-called φ-operator φ is connected - introduced in Pedrycz (1982), but denoted ψ there - which may be characterized through

$$a \varphi b =_{def} \sup \{ z \mid a \, t \, z \leq b \}$$

for all $a,b \in [0,1]$. We avoid to mention the reference to the t-norm in the denotation of φ-operator. With each t-norm exactly one φ-operator is connected, and the t-norm referred to will always be clear from context. Those φ-operators will be used as truth functions for implication operators $\rightarrow_t$;

here the connection with t-norm is made obvious by notation.

With those generalized connectives we define as in Gottwald (1983) generalized inclusion relations for fuzzy sets A,B by

$$A \subseteq_t B =_{def} \forall x(x \in A \rightarrow_t x \in B)$$

and furthermore generalized identity relations too by

$$A =_t B =_{def} A \subseteq_t B \,\&_t\, B \subseteq_t A .$$

Both these generalized relations are many valued, i.e. fuzzy ones which assume truth values in the real interval $[0,1]$. In corresponding way, for each binary operation $*$ within the universe of discourse their extension to fuzzy sets on this universe may be realized through any t-norm **t** as

$$A *_t B =_{def} \{ a * b \,\|\, a \in A \,\&_t\, b \in B \} .$$

With this notation an immediate generalization of the main result from Gottwald (1983) can be stated as:

$$\models \exists X(A *_t X =_t B) \longleftrightarrow_t A *_t (B \tilde{*}_t A) =_t B \qquad (1)$$

where yet we have to suppose that the t-norm **t** is lower semicontinuous, and where $\longleftrightarrow_t$, $\tilde{*}_t$ are defined by

$$p \longleftrightarrow_t q =_{def} (p \rightarrow_t q) \,\&_t\, (q \rightarrow_t p) ,$$

$$B \tilde{*}_t A =_{def} \{ y \,\|\, \forall x(x \in A \rightarrow_t x * y \in B) \} .$$

The second type of fuzzy equations we are interested in sometimes is referred to as fuzzy relational equations and written $A \circ R = B$ with a fuzzy relation R and fuzzy sets A,B and, in our notation,

$$A \circ R =_{def} \{ y \,\|\, \exists x(x \in A \,\&_t\, (x,y) \in R) \} ,$$

t any t-norm or simply **t** = min. Because of the analogy of this construction with full image of a set under a relation we prefer to write R"A for $A \circ R$, i.e. we consider the equations

$$R''A = B$$

with unknown fuzzy relation R, again here avoiding explicite reference to the t-norm involved.

For this new type of fuzzy equations the corresponding result to (1) is that

$$\models \exists R(R''A =_t B) \longleftrightarrow_t (A \circledast B)''A =_t B \qquad (2)$$

for all lower semicontinuous t-norms **t** and

$$A \circledast B =_{def} \{(x,y) \parallel x \varepsilon A \longrightarrow_t y \varepsilon B\} .$$

Because one has for lower semicontinuous t-norms t that it holds true that for any formulas H_1, H_2

$$\models H_1 \longleftrightarrow_t H_2 \quad \text{iff} \quad [H_1] = [H_2]$$

i.e. iff both formulas have the same truth value, our results (1) and (2) give a characterization of the truth value of $\exists X(A *_t X =_t B)$ resp. of $\exists R(R''A =_t B)$, i.e. both (1) and (2) give information on the existence of solutions of fuzzy equation $A *_t X = B$ resp. of fuzzy equation $R''A = B$. But, we get not only information on the existence of solutions, also in case there does not exist a solution our results characterize "best possible" approximate solutions, which depends of the fact that the function

$$\rho_t(A,B) =_{def} 1 - [A =_t B]$$

is a metric for each lower semicontinuous t-norm t which is not smaller than the algebraic product t_m defined as

$$a \; t_m \; b =_{def} \max(0, a+b-1) .$$

Now, let us consider systems of fuzzy equations of both types

$$R''A_i = B_i , \qquad i = 1,\dots,N \qquad (3)$$

$$A_i *_t X = B_i , \qquad i = 1,\dots,N . \qquad (4)$$

As in formulas (1),(2) we could write down the results concerning solvability of such systems quite directly, but to get shorter and easier to read formulas we introduce some additional abbreviations. With

$$\mathop{\mathbf{T}}_{i=1}^{n} H_i =_{def} H_1 \&_t H_2 \&_t \dots \&_t H_n$$

as the finite iteration of our t-norm conjunction $\&_t$ we put

$$\text{solv}_t =_{def} \exists R(\mathop{\mathbf{T}}_{i=1}^{N}(R''A_i =_t B_i)) , \qquad (5)$$

$$\mathrm{solv}'_t =_{def} \exists X\Big(\mathop{\mathbf{T}}_{i=1}^{N} (A_i *_t X =_t B_i)\Big) \quad . \tag{6}$$

Unfortunately, in general we are not able to give full characterizations of (the truth values of) formulas (5),(6) but only upper and lower bounds for the truth values, which again may be written as implicational formulas because of

$$\models H_1 \longrightarrow_t H_2 \qquad \text{iff} \qquad [H_1] \leq [H_2]$$

for lower semicontinuous t-norms **t**. As those upper and lower bounds are given by the same formulas we introduce again two abbreviations:

$$\mathrm{char}_t =_{def} \mathop{\mathbf{T}}_{i=1}^{N} \Big(\Big(\bigcap_{j=1}^{N} (A_j \circledS B_j) \Big) " A_i =_t B_i \Big) \quad ,$$

$$\mathrm{char}'_t =_{def} \mathop{\mathbf{T}}_{i=1}^{N} \Big(A_i *_t \bigcap_{j=1}^{N} (B_j \tilde{*}_t A_j) =_t B_i \Big) \quad .$$

Now we can present the results in short formulas. The lower bounds are given through the facts that there hold true - for lower semicontinuous **t** always -

$$\models \mathrm{char}_t \longrightarrow_t \mathrm{solv}_t \quad ,$$

$$\models \mathrm{char}'_t \longrightarrow_t \mathrm{solv}'_t \quad .$$

The upper bounds are harder to describe. For the general case it is simpler not to give directly an upper bound for (the truth value of) formulas solv_t, solv'_t but an upper bound for a suitable iteration. The results are

$$\models \mathop{\mathbf{T}}_{i=1}^{N} \mathrm{solv}_t \longrightarrow_t \mathrm{char}_t \quad ,$$

$$\models \mathop{\mathbf{T}}_{i=1}^{N} \mathrm{solv}'_t \longrightarrow_t \mathrm{char}'_t \quad .$$

As corollaries we get that system (3) of fuzzy "relational" equations has a solution iff $\bigcap_{i=1}^{N} (A_i \circledS B_i)$ is a solution, and each solution of (3) is a subset of this fuzzy relation; and we get analogously that system (4) of fuzzy "arithmetical" equations has a solution iff $\bigcap_{i=1}^{N} (B_i \tilde{*}_t A_i)$ is a solution, and each solution of (4) is a subset of this fuzzy set.

Furthermore, for $t = \min$ we get the full characterizations from idempotency of the iteration of the min-operator:

$$\models \mathrm{solv}_{\min} \longleftrightarrow_{\min} \mathrm{char}_{\min} ,$$

$$\models \mathrm{solv}'_{\min} \longleftrightarrow_{\min} \mathrm{char}'_{\min} .$$

The full proofs of the results that have been sketched here and of corresponding ones for other types of fuzzy equations are given in Gottwald (198x). And a slight extension proves useful for non-trivial applications to the design of fuzzy controllers; cf. Gottwald/Pedrycz (198x).

References

Gottwald,S. (1983). Generalization of some results of Elie Sanchez. BUSEFAL, no. 16, 54-60.

Gottwald,S. (1984). On the existence of solutions of systems of fuzzy equations. Fuzzy Sets Syst. 12, 301-302.

Gottwald,S. (198x). Characterizations of the solvability of fuzzy equations. (submitted to Elektron. Informationsverarb. Kybernet.)

Gottwald,S. and W.Pedrycz (198x). Solvability of fuzzy relational equations and manipulation of fuzzy data. (to be submitted to Fuzzy Sets Syst.)

Pedrycz,W. (1982). Fuzzy relational equations with triangular norms and their resolutions. BUSEFAL, no. 11, 24-32.

Sanchez,E. (1984). Solution of fuzzy equations with extended operations. Fuzzy Sets Syst. 12, 237-248.