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Fuzzy permutation with lower solution of Fuzzy relation equation

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Abstract

The lower solutions of Fuzzy relation equation are discussed; The reflexivable permutation E, operator $\hat{r}$ and the operation of E with $\hat{r}$ are definied in this paper. Fuzzy permutation p represented in [1], [2], it is to find the lower solutions of Fuzzy relation equation and their necessary and sufficient condition, thus a complete method for seeking lower solutions is established. If this method is tabulated, it is convenience to find all lower solutions without repetition.

A Reflexivable permutation

Let $U=\{1,2,\cdots,n\}$, $V=\{1,2,\cdots,m\}$. A mapping p, corresponding each point of U to a element of $\mathcal{F}(V)$ (Fuzzy power set), is called a Fuzzy permutation and denoted by

$$p=\begin{pmatrix} 1 & 2 & \cdots & n \\ \underset{\sim}{R}_1 & \underset{\sim}{R}_2 & \cdots & \underset{\sim}{R}_n \end{pmatrix}.$$

where $\underset{\sim}{R}_i = r_{i1}/1 + r_{i2}/2 + \cdots + r_{im}/m \in \mathcal{F}(V)$, $i \in U$. For convenience, we rewrite $\underset{\sim}{R}_i$ as the form

$$\underset{\sim}{R}_i=\left(\frac{r_{i1}}{1}, \frac{r_{i2}}{2}, \cdots, \frac{r_{im}}{m}\right)$$

If $U=V$ and $\underset{\sim}{R}_i$ is a Fuzzy namber, then p is called a regular Fuzzy permutation.

$\underset{\sim}{R}_i$ is also called the Fuzzy image of i under Fuzzy permutation p, which is denoted as $p(i)=\underset{\sim}{R}_i$. Furthermore, we take $p(i_1, i_2), \cdots, p(U)$ as $\{\underset{\sim}{R}_{i_1}, \underset{\sim}{R}_{i_2}\}, \cdots, \{\underset{\sim}{R}_1, \underset{\sim}{R}_2, \cdots, \underset{\sim}{R}_n\}$ respectively. $\underset{\sim}{R}_i(j)$ denote the degree of point i permuting by point j.

Let $t_{ji}=r_{ij}$, $\underset{\sim}{T}_j=\left(\frac{t_{1j}}{1}, \frac{t_{2j}}{2}, \cdots, \frac{t_{nj}}{n}\right) \in \mathcal{F}(U)$, then $p^{-1}=\begin{pmatrix} 1 & 2 & \cdots & m \\ \underset{\sim}{T}_1 & \underset{\sim}{T}_2 & \cdots & \underset{\sim}{T}_m \end{pmatrix}$ is called the transposed permutation.

For the multiplication operation "$\circ$" defined in [1], we have generally $p \circ p^{-1} \neq p^{-1} \circ p$.

Definition 1 Let $\underset{\sim}{R}_i = e_{ii}/i$, $e_{ii} \in [0,1]$

$$E=\begin{pmatrix} 1 & 2 & \cdots & n \\ \left(\frac{e_{11}}{1}\right) & \left(\frac{e_{22}}{2}\right) & \cdots & \left(\frac{e_{nn}}{n}\right) \end{pmatrix}$$

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is called a Fuzzy reflexivable permutation. or, reflexivable permutation.

When $\forall i \in U$, $e_{ii} \equiv 1$, E is the complete reflexivable permutation, i.e. the unit permutation.

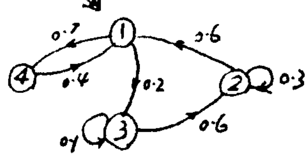
When $\forall i \in U$, $e_i e_i \equiv 0$, E is the inverse reflexivable permutation.

Under the finite universe of discourse (or universe), Fuzzy relation is representationed by the Fuzzy matrix、Fuzzy graph and Fuzzy permutation, therefore, we can establish their corresponding relation

Example 1 Let $\underset{\sim}{R} = \frac{0.2}{(1,3)} + \frac{0.7}{(1,4)} + \frac{0.6}{(2,1)} + \frac{0.3}{(2,2)} + \frac{0.6}{(3,2)} + \frac{0.1}{(3,3)} + \frac{0.4}{(4,1)}$ be Fuzzy relation. The corresponding relation of the Fuzzy matrix、Fuzzy graph and the Fuzzy permutation is

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ (\frac{0.2}{3}, \frac{0.7}{4}) & (\frac{0.3}{2}, \frac{0.6}{1}) & (\frac{0.1}{3}, \frac{0.6}{2}) & (\frac{0.4}{1}) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0.6 & 0 & 0.4 \\ 0 & 0.3 & 0.6 & 0 \\ 0.2 & 0 & 0.1 & 0 \\ 0.7 & 0 & 0 & 0 \end{pmatrix} \Longleftrightarrow$$



(fig. 1)

The difference of general permutation and Fuzzy permutation is that the general permutation corresponds point to point, where Fuzzy permutation corresponds point to Fuzzy Set.

In example 1, p replaces point "1" by $\underset{\sim}{R}_1 = (\frac{0.2}{3}, \frac{0.7}{4})$, points "3"、"4" in $supp(\underset{\sim}{R}_1) = \{3, 4\}$ by $\underset{\sim}{R}_3 = (\frac{0.1}{3}, \frac{0.6}{2})$ and $\underset{\sim}{R}_4 = (\frac{0.4}{1})$ respectively, points "1"、"2" and "3" in $Supp(\underset{\sim}{R}_3) \cup Supp(\underset{\sim}{R}_4)$ by $\underset{\sim}{R}_1 = (\frac{0.2}{3}, \frac{0.7}{4})$, $\underset{\sim}{R}_2 = (\frac{0.3}{2}, \frac{0.6}{1})$ and $\underset{\sim}{R}_3 = (\frac{0.1}{3}, \frac{0.6}{2})$ respectively. and so on.

B Characteristic Fuzzy set and $\hat{\delta}$ operator

In [2], The Fuzzy relation equation $A \circ X = B$ has been transform into iso-solution equation $A^* \circ X = B$. Where A^* is iso-solution matrix of A and $a^*_{ji} \in \{0, b_j\}$.

Example 2 The iso-solution matrix of Fuzzy relation equation

$$\begin{pmatrix} 0.8 & 0.9 & 0.7 & 0.5 & 0.4 & 0.1 \\ 0.8 & 0.7 & 0.8 & 0.7 & 0.6 & 0.6 \\ 0.3 & 0.8 & 1 & 0.9 & 0.6 & 0.9 \\ 0.6 & 0.6 & 0.7 & 0.8 & 0.8 & 0.8 \\ 0.4 & 0.3 & 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.2 & 0.4 & 0.3 & 0.6 & 0.5 \\ 0.2 & 0.4 & 0.3 & 0.3 & 0.5 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.1 & 0.4 & 0.4 \\ 0.3 & 0.1 & 0 & 0.3 & 0.1 & 0.3 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \\ 0.8 \\ 0.7 \\ 0.5 \\ 0.5 \\ 0.4 \\ 0.4 \\ 0.3 \end{pmatrix} \qquad (1)$$

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$$A^* = \begin{pmatrix} 0.8 & 0.8 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0.8 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.4 \\ 0.3 & 0 & 0 & 0.3 & 0 & 0.3 \end{pmatrix} \tag{2}$$

In the rest of the paper. we suppose

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m$$

and $b_1 = b_2 = \cdots = b_{l_1} = \lambda_1$, $b_{l_1+1} = \cdots = b_{l_2} = \lambda_2$, $\cdots$, $b_{l_{t-1}} = \cdots = b_m = b_{l_t} = \lambda_t$ whenever we discuss iso-solution equation.

The element a^*_{ji} of iso-solution matrix A^* is equal to b_j or greater to b_j. $p(i) = \underset{\sim}{R}_i$ is Fuzzy set which is consisted of all non-zero element of i-th column of A^*. Obviously, Fuzzy image of i regarded as Fuzzy vector.

Definition 2 Let A^* corresponding to Fuzzy permutation p, and $\lambda \in [0,1]$, $\exists j \in V$. Such that

$$\underset{\sim}{R}_i(j) = \lambda\ ,$$

$$\bigcup_{i_l \in I} \{\underset{\sim}{R}_{i_l} \mid \underset{\sim}{R}_{i_l}(j) = \lambda\}$$

is called the lower bound Fuzzy set of U for λ, in brief, λ-lower Fuzzy set, and

$$I = \{i_1, i_2, \cdots, i_k\} \subset U$$

is called the λ-lower index set

Obvious, $(\underset{\sim}{R}_{i_1} \cup \underset{\sim}{R}_{i_2} \cup \cdots \cup \underset{\sim}{R}_{i_k})_\lambda = \{j_1, j_2, \cdots, j_r\} = J \subset V$, J is called the λ-lower cuts set.

In example 2, The A^* corresponds to p:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \left(\frac{0.8}{1}, \frac{0.8}{2}, \frac{0.5}{6}, \frac{0.3}{9}\right) & \left(\frac{0.8}{1}, \frac{0.8}{3}, \frac{0.4}{7}\right) & \left(\frac{0.8}{2}, \frac{0.8}{3}, \frac{0.7}{4}, \frac{0.5}{5}\right) & \left(\frac{0.7}{4}, \frac{0.3}{9}\right) & \left(\frac{0.4}{7}, \frac{0.4}{8}\right) & \left(\frac{0.5}{5}, \frac{0.5}{6}, \frac{0.4}{8}, \frac{0.3}{9}\right) \end{pmatrix}$$

Let $\lambda = 0.8$, there exist $j = 1, 2, 3 \in V$, such that

$$\underset{\sim}{R}_1(1) = \underset{\sim}{R}_2(3) = \underset{\sim}{R}_3(2) = 0.8$$

so $\left(\frac{0.8}{1}, \frac{0.8}{2}, \frac{0.5}{6}, \frac{0.3}{9}\right) \cup \left(\frac{0.8}{1}, \frac{0.8}{3}, \frac{0.4}{7}\right) \cup \left(\frac{0.8}{2}, \frac{0.8}{3}, \frac{0.7}{4}, \frac{0.5}{5}\right)$ is λ-lower bound Fuzzy set, and $I = \{1, 2, 3\} \subset U$ is λ-lower bound index set.

Because $(\underset{\sim}{R}_1 \cup \underset{\sim}{R}_2 \cup \underset{\sim}{R}_3)_{0.8} = \{1, 2, 3\} \subset V = \{1, 2, \cdots, 9\}$, so λ-lower bound cuts is $J = \{1, 2, 3\}$.

Definition 3 Suppose that there is λ-lower bound index set $I = \{i_1, i_2, \cdots, i_k\}$, if $\exists I_t \subset I$ $(I_t = \{i_{t_1}, i_{t_2}, \cdots, i_{t_r}\})$, such that

$$(\underset{\sim}{R}_{i_{t_1}} \cup \underset{\sim}{R}_{i_{t_2}} \cup \cdots \cup \underset{\sim}{R}_{i_{t_r}})_\lambda = J\ .$$

but $\forall I_s \subsetneqq I_t \quad (s \geq 1)$, $I_s = \{i_{s_1}, i_{s_2}, \cdots, i_{s_l}\}$, then

$$(\underset{\sim}{R}_{i_{s_1}} \cup \underset{\sim}{R}_{i_{s_2}} \cup \cdots \underset{\sim}{R}_{i_{s_l}})_\lambda \subsetneqq J$$

thus $(\underset{\sim}{R}_{i_{t_1}} \cup \underset{\sim}{R}_{i_{t_2}} \cup \cdots \cup \underset{\sim}{R}_{i_{t_r}})$ is called the Characteristic Fuzzy set, and I_t is called the Characteristic index set. Without any confusion the characteristic index set be represented by $\{i_1, i_2, \cdots, i_k\}$ and characteristic Fuzzy set by $\underset{\sim}{R}_{i_1} \cup \underset{\sim}{R}_{i_2} \cup \cdots \cup \underset{\sim}{R}_{i_k}$.

Obviously, the characteristic index set is chosen by means of more than one ways in I, and when $t_1 \neq t_2$, then $I_{t_1} \not\subset I_{t_2}$ or $I_{t_2} \not\subset I_{t_1}$, denotely I_t^c by $I \setminus I_t$.

Definition 4 Let Characteristic Fuzzy Set be $\underset{\sim}{R}_{i_1} \cup \underset{\sim}{R}_{i_2} \cup \cdots \cup \underset{\sim}{R}_{i_k}$ and $\lambda \in [0,1]$. Definiting reflexivable permutation of p is

$$E = \begin{pmatrix} 1 & 2 & \cdots & m \\ \underset{\sim}{E}_1 & \underset{\sim}{E}_2 & \cdots & \underset{\sim}{E}_n \end{pmatrix}, \quad \underset{\sim}{E}_j = \frac{e_{ij}}{j}, \ j \in V$$

for characteristic Fuzzy set, where

$$e_{ij} = \begin{cases} 0, & j \text{ satisfing } \ 0 \leq \underset{\sim}{R}_i(j) < \lambda \quad (i \in I_t). \\ \lambda, & \text{other } j. \end{cases}$$

For example, we take out characteristic index set $I_1 = \{1, 2\}$ in p, thus

$$E_{I_1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ (\frac{0.8}{1}) & (\frac{0.8}{2}) & (\frac{0.8}{3}) & (\frac{0.8}{4}) & (\frac{0.8}{5}) & (\frac{0}{6}) & (\frac{0}{7}) & (\frac{0.8}{8}) & (\frac{0}{9}) \end{pmatrix}$$

Definition 5 Let reflexivable permutation of Fuzzy permutation p for characteristic Fuzzy set be E, we definite the operation $\hat{\gamma}$:

$$p' = E \hat{\gamma} p = \begin{pmatrix} 1 & 2 & \cdots & m \\ \underset{\sim}{E}_1 & \underset{\sim}{E}_2 & \cdots & \underset{\sim}{E}_m \end{pmatrix} \hat{\gamma} \begin{pmatrix} 1 & 2 & \cdots & n \\ \underset{\sim}{R}_1 & \underset{\sim}{R}_2 & \cdots & \underset{\sim}{R}_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \underset{\sim}{G}_1 & \underset{\sim}{G}_2 & \cdots & \underset{\sim}{G}_n \end{pmatrix}$$

where $\underset{\sim}{G}_i = \left(\frac{g_{i1}}{1}, \frac{g_{i2}}{2}, \cdots, \frac{g_{im}}{m}\right)$, $g_{ij} = \begin{cases} e_{ij} \gamma r_{ij}, & (i \in I_t). \\ e_{ij} \dot{\gamma} r_{ij}, & (\text{otherwise}). \end{cases}$

and γ, $\dot{\gamma}$ are definited respectively by

$$a \gamma b \triangleq \begin{cases} b, & (a \geq b), \\ 0, & (a < b); \end{cases} \qquad a \dot{\gamma} b \triangleq \begin{cases} b, & (a > b), \\ 0, & (a \leq b). \end{cases}$$

For all $\lambda \in [0,1]$, the definitions 4 and definition 5 repeatedly apply to p, and let $\underset{\sim}{R}_i^{(t)}(j) = x_i$, then

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \underset{\sim}{R}_1^{(t)}(j) \\ \underset{\sim}{R}_2^{(t)}(j) \\ \vdots \\ \underset{\sim}{R}_n^{(t)}(j) \end{pmatrix} \tag{3}$$

is a lower solution of Fuzzy relation equation. In example 2, $(0.8, 0.8, 0.7, 0, 0, 0.5)^T$ is a lower solution

C Necessary and sufficient condition of Fuzzy relation equation have a lower solution

Theorem 1 Suppose that $A^* \circ X = B$, $A^* \to p$. Fuzzy relation equation has solution iff

$$\text{supp}\{\underset{\sim}{R}_1, \underset{\sim}{R}_2, \cdots, \underset{\sim}{R}_n\} = \text{Supp}[p(U)] = V.$$

Proof Because $A^* \rightleftarrows p$, the $\underset{\sim}{R_1}, \underset{\sim}{R_2}, \cdots, \underset{\sim}{R_n}$ corresponde respectively to 1-th, 2-th, …, n-th column matrix of A^*. Obviously, row order number set of A^* is V, Column order number set of A^* is U.

Necessity Because $A^* \circ X = B$ has solution, then

$$\begin{cases} \bigvee_{i=1}^{m} (a_{ji}^* \wedge x_i) \le b_j & (\forall j \in V), \\ (a_{j_{i_0} i_0}^* \wedge x_{i_0}) = b_j & (\forall j \in V, \ \exists i_0 \in U). \end{cases}$$

So that, there exists non-zero element a_{ji}^* of A^* at least in each row of A^*; and because $a_{ji}^* = r_{ij}$ $(= p_i(j))$, i.e. non-zero r_{ij}, then there exist non-zero element r_{ij} in $\underset{\sim}{R_{i_1}}$ or $\underset{\sim}{R_{i_2}}$, …, or $\underset{\sim}{R_n}$, i.e. there are non-zero elements in

$$\underset{\sim}{R_1} \cup \underset{\sim}{R_2} \cup \cdots \cup \underset{\sim}{R_n} .$$

and j is any element of V, so that

$$Supp\{\underset{\sim}{R_1} \cup \underset{\sim}{R_2} \cup \cdots \cup \underset{\sim}{R_n}\} = Supp(p(U)) = V$$

Sufficiency Suppose that $A^* \circ X = B$, $A^* \rightleftarrows p$, then $a_{ji}^* \in \{0, b_j\}$.

Let $Supp(p(U)) = V$, then there are m non-zero r_{ij} in $\underset{\sim}{R_1} \cup \underset{\sim}{R_2} \cup \cdots \cup \underset{\sim}{R_n}$ at least.

If $$a_{1i_1}^* = b_1, \quad a_{2i_2}^* = b_2, \quad \cdots, \quad a_{mi_n}^* = b_m,$$

obviously $$(a_{j1}^* \wedge x_1) \vee (a_{j2}^* \wedge x_2) \vee \cdots \vee (a_{jn}^* \wedge x_n) \le b_j \qquad (j = 1, 2, \cdots m)$$

therefore, for each j, there is at least one element of I, say i_0, then

$$a_{j_{i_0} i_0}^* \wedge x_{i_0} = b_j$$

thus $$A^* \circ X = B.$$

Corollary 1 $Supp[p^{-1}(j)] \ne \phi$

Definition 6 Suppose that $A^* \circ X = B$, for each row (colum) of A^* operated "$\vee$" according to column (row) a_{ji}^*, then A^* becomes the column (row) matrix, its called collumn (row) projection. It denoted by

$$\bigvee_{i \in U} A^* \qquad (\bigvee_{j \in V} A^*)$$

Theorem 2. Suppose that there is a solution in $A^* \circ X = B$, then

1° $\bigvee_{i \in U} A^* = B$.

2° $A^* \circ (\bigvee_{j \in V} A^*)^T = B$ (T representing transpose operation).

Proof 1° Because $A^* \circ X = B$ has sulotion, thus by the result of corollary 1 of theorem 1, $\forall j \in V$, we obtain

$$Supp[p^{-1}(j)] \ne \phi \qquad \text{(each row nonempty set of } A^*).$$

Because $a_{ji}^* \in \{0, b_j\}$, thus, for any j such that

$$\max\{a_{j1}^*, a_{j2}^*, \cdots, a_{jn}^*\} = b_j .$$

Because j is any element, then

$$\bigvee_{i \in U} A^* = B .$$

2° If $A^* \circ X = B$ has a salution X,

$$\begin{cases} (a_{11}^* \wedge x_1) \vee \cdots \vee (a_{1i}^* \wedge x_i) \vee \cdots \vee (a_{1n}^* \wedge x_n) = b_1 \\ (a_{21}^* \wedge x_1) \vee \cdots \vee (a_{2i}^* \wedge x_i) \vee \cdots \vee (a_{2n}^* \wedge x_n) = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ (a_{m1}^* \wedge x_1) \vee \cdots \vee (a_{mi}^* \wedge x_i) \vee \cdots \vee (a_{mn}^* \wedge x_n) = b_m \end{cases} \tag{4}$$

thus $\begin{cases} a_{ji}^{*} \wedge x_i \le b_j , & (\forall j \in V, \ \forall i \in U) . \\ a_{j i_0}^{*} \wedge x_{i_0} = b_j , & (\forall j \in V, \ \exists i_0 \in U) . \end{cases}$

Obviously, $a_{j i_0}^{*} \geq b_j$, $x_{i_0} \geq b_j$ and $a_{ji} \in \{0, b_j\}$. thus $a_{j i_0} = b_j$, then
$\forall j \in V$, $\exists i_0 \in U$ such that $x_{i_0} \geq b_j = a_{j i_0}^{*}$.

Let $x_{i_1}' = \max\{a_{1i_1}^{*}, a_{2i_1}^{*}, \cdots, a_{mi_1}^{*}\}$ $(\forall j \in V, \exists i_1 \in U)$

Because $supp(p^{-1}(j)) \neq \phi$ $(\forall j \in V)$. for i_1-th column there is no harm in leting

$$a_{1i_1}^{*} = b_1 .$$

therefore $x_{i_1}' \geq \max\{a_{1i_1}^{*}, a_{2i_1}^{*}, \cdots, a_{mi_1}^{*}\} = a_{1i_1}^{*} = b_1$

thus we replaces x_{i_1} of first equation of (4) by x_{i_1}' , then

$$(a_{11}^{*} \wedge x_1) \vee \cdots \vee (a_{1i_1}^{*} \wedge x_{i_1}') \vee \cdots \vee (a_{1n}^{*} \wedge x_n) = (a_{11}^{*} \wedge x_1) \vee \cdots \vee (b_1 \wedge x_{i_1}') \vee \cdots \vee (a_{1n}^{*} \wedge x_n) = b_1$$

Similarly, for i_2, following formula holds

$$x_{i_2}' \geq \max\{a_{1i_2}^{*}, a_{2i_2}^{*}, \cdots, a_{mi_2}^{*}\} = a_{2i_2}^{*} = b_2 \leq b_1 ,$$

Then we replace x_{i_1} together with x_{i_2} by x_{i_1}', x_{i_2}' respectively in the first two equations of (4), therefore

$$(a_{11}^{*} \wedge x_1) \vee \cdots \vee (a_{1i_1}^{*} \wedge x_{i_1}') \vee \cdots \vee (a_{1i_2}^{*} \wedge x_{i_2}') \vee \cdots \vee (a_{1n}^{*} \wedge x_n) = b_1 ,$$
$$(a_{21}^{*} \wedge x_1) \vee \cdots \vee (a_{2i_1}^{*} \wedge x_{i_1}') \vee \cdots \vee (a_{2i_2}^{*} \wedge x_{i_2}') \vee \cdots \vee (a_{2n}^{*} \wedge x_n) = b_2 .$$

Similarly. $\forall i \in U$, we solve $x_1', x_2', \cdots, x_n'$, let $X' = (x_1', x_2', \cdots, x_n')^T$, then

$$A^{*} \circ X' = B .$$

where

$$X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} \max\{a_{11}^{*}, a_{21}^{*}, \cdots, a_{m1}^{*}\} \\ \max\{a_{21}^{*}, a_{22}^{*}, \cdots, a_{m2}^{*}\} \\ \cdots\cdots\cdots\cdots \\ \max\{a_{m1}^{*}, a_{m2}^{*}, \cdots, a_{mn}^{*}\} \end{pmatrix} = (\bigvee_{j \in V} A^{*})^T$$

i.e. $A^{*} \circ (\bigvee_{j \in V} A^{*})^T = B$.

Because $A^{*} \rightleftarrows p$, evidently

$$\bigvee_{j \in V} A^{*} = (\bigvee_{j \in V} p(1), \bigvee_{j \in V} p(2), \cdots, \bigvee_{j \in V} p(n))$$
$$= (\bigvee_{j \in V} \underset{\sim}{R}_1, \bigvee_{j \in V} \underset{\sim}{R}_2, \cdots, \bigvee_{j \in V} \underset{\sim}{R}_n)$$

$(\bigvee_{j \in V} p(1), \cdots, \bigvee_{j \in V} p(n))$ denoted by $\bigvee_{j \in V} p$, then

Corollary 2 Suppose that there is a solution in $A^{*} \circ X = B$, then $A^{*} \circ (\bigvee_{j \in V} p)^T = B$

For convenience, the p is resolved by the union of Fuzzy permutation, for instance

$$p = \begin{pmatrix} 1 & 2 & 3 \\ (\frac{0.8}{1}, \frac{0.8}{2}, \frac{0.5}{6}, \frac{0.3}{9}) & (\frac{0.8}{1}, \frac{0.8}{3}, \frac{0.4}{7}) & (\frac{0.5}{5}, \frac{0.5}{6}, \frac{0.4}{8}) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 \\ (\frac{0.8}{1}, \frac{0.8}{2}) & (\frac{0.4}{7}) & (\frac{0.5}{5}, \frac{0.5}{6}) \end{pmatrix} \cup \begin{pmatrix} 1 & 2 & 3 \\ (\frac{0.5}{6}, \frac{0.3}{9}) & (\frac{0.8}{1}, \frac{0.8}{3}) & (\frac{0.4}{8}) \end{pmatrix}$$

then, we have

Definition 7 Let p be Fuzzy permutation and

$$p = p^1 \cup p^2 \cup \cdots \cup p^K . \quad \text{where} \quad p^l = \begin{pmatrix} 1 & 2 & \cdots & n \\ \underset{\sim}{R}_1^l & \underset{\sim}{R}_2^l & \cdots & \underset{\sim}{R}_n^l \end{pmatrix} \quad (1 \le l \le K)$$

and $\underset{\sim}{R}_i = \underset{\sim}{R}_i^1 \cup \underset{\sim}{R}_i^2 \cup \cdots \cup \underset{\sim}{R}_i^l$ $(1 \le i \le n)$. $p^1 \cup p^2 \cup \cdots \cup p^K$ is called resolution of p .

By means of this representation of p we obtain

$$A^{*}\circ(\bigvee_{j\in V}p)^{T}=A^{*}\circ(\bigvee_{j\in V}(p^{1}\cup p^{2}\cup\cdots\cup p^{k}))^{T}$$
$$=A^{*}\circ(\bigvee_{j\in V}p^{1})^{T}\cup A^{*}\circ(\bigvee_{j\in V}p^{2})^{T}\cup\cdots\cup A^{*}\circ(\bigvee_{j\in V}p^{k})^{T}=B,$$

evidently $$A^{*}\circ(\bigvee_{j\in V}p^{k})^{T}\subset B \quad (k\geqslant 1)$$

According to definition 7 we obtain

Theorem 3 If $A^{*}\circ(\bigvee_{j\in V}p)^{T}=B$, then there exists certain p^{t} ($p^{t}\subset p)^{(1)}$ such that

$$A^{*}\circ(\bigvee_{j\in V}p^{t})^{T}=B \qquad (1\leq t\leq k)$$

Definition 8 If $A^{*}\circ(\bigvee_{j\in V}p^{t})^{T}=B$, $\forall p^{t_i}\subsetneqq p^{t}$, $A^{*}\circ(\bigvee_{j\in V}p^{t_i})^{T}\subsetneqq B$, then $(\bigvee_{j\in V}p^{t})^{T}$ is called lower solution of $A^{*}\circ X=B$.

Theorem 4 Let X_0 be lower solution of $A^{*}\circ X=B$, then $\exists x_i\in X_0$, such that

$$x_i=\max\{b_1,b_2,\cdots,b_m\}$$

where x_i is component of X_0

Proof If $(\bigvee_{j\in V}p^{t})^{T}$ is lower solution of $A^{*}\circ X=B$, then

$$A^{*}\circ(\bigvee_{j\in V}p^{t})^{T}=B$$

thus $$(\bigvee_{j\in V}p^{t})=(\bigvee_{j\in V}\underset{\sim}{R}_1^{t},\bigvee_{j\in V}\underset{\sim}{R}_2^{t},\cdots,\bigvee_{j\in V}\underset{\sim}{R}_n^{t})=(x_1,x_2,\cdots,x_n)$$

there is no narm in that let maximum component of $(x_1,x_2,\cdots,x_n)$ be x_i. Certainly there is

$$\bigvee_{j\in V}\underset{\sim}{R}_i^{t}=\max\{a_{1i}^{*},a_{2i}^{*},\cdots,a_{mi}^{*}\} \qquad (i\in U)$$
$$=x_i=b_1$$

This theorem represents that there must exist maximum component at least in lower solution, the maximum component is b_1.

For convenience, we lead into following sign:

(1) Maximum component of solution of $A^{*}\circ X=B$ denoted by $\langle\langle X\rangle\rangle$, for instance, $\langle\langle(0.8,0.8,0.7,0,0.4,0)\rangle\rangle\triangleq(0.8,0.8,0,0,0,0)$

(2) Let $x_{i_1},x_{i_2},\cdots,x_{i_k}$ be k components of X, we have

$$(\{x_{i_k}\})\triangleq(0,\cdots,0,x_{i_1},0,\cdots,0,x_{i_2},0,\cdots,0,x_{i_k},0,\cdots,0)$$

for instence, $(\{0.8_1,0.4_5\})=(0.8,0,0,0,0.4,0)$.

Definition 9 If $A^{*}\circ X=B$, and $b_{j_1}=b_{j_2}=\cdots=b_{j_t}=\lambda_t$, we take out

$$\begin{cases}(a_{j_1 1}\wedge x_1)\vee(a_{j_1 2}\wedge x_2)\vee\cdots\vee(a_{j_1 n}\wedge x_n)=b_{j_1}\\(a_{j_2 1}\wedge x_1)\vee(a_{j_2 2}\wedge x_2)\vee\cdots\vee(a_{j_2 n}\wedge x_n)=b_{j_2}\\\cdots\cdots\cdots\cdots\cdots\cdots\\(a_{j_t 1}\wedge x_1)\vee(a_{j_t 2}\wedge x_2)\vee\cdots\vee(a_{j_t n}\wedge x_n)=b_{j_t}\end{cases}\qquad(5)$$

from $A^{*}\circ X=B$. (5) is called the λ_t-level lumped Fuzzy relation equation, and denoted by $A^{*}_{J(\lambda_t)}\circ X=B_{J(\lambda_t)}$, in brief, λ_t-lumped equation. Evidently, we have

Theorem 5 The solutions of $A^{*}\circ X=B$ is also solutions of λ-lumped equation.

Theorem 6 If $A^{*}\circ X=B$ has solutions, maximum component of lower solution are $x_{i_1},x_{i_2},\cdots,x_{i_k}$ iff $I_t=\{i_1,i_2,\cdots,i_k\}$ is characteristic index set.

Proof. Let maximum component of lower solution of $A^{*}\circ X=B$ be $x_{i_1}=x_{i_2}=\cdots=x_{i_k}=\lambda_1$. Thus, we can take one λ_1-lumped equation with j rows from $A^{*}\circ X=B$.

$$A^{*}_{J(\lambda_1)}\circ X=B_{J(\lambda_1)}$$

where $J(\lambda_1)=\{1,2,\cdots,j\}$ represents the set definited by λ_1. Evidently, $A^{*}_{J(\lambda_1)}$ is a $j\times n$ matrix, and $A^{*}_{J(\lambda_1)}$ correspond to $p_{J(\lambda_1)}$, lower solution of $A^{*}\circ X=B$ is also lower solution of $A^{*}_{J(\lambda_1)}\circ X=B_{J(\lambda_1)}$. Then, there are solutions of $A^{*}_{J(\lambda_1)}\circ X=B_{J(\lambda_1)}$, and by theorem 1,

$$supp[\,p_{J(\lambda_1)}(U)]=J(\lambda_1)=\{1,2,\cdots,j\}$$

holds. There is no harm in letting

$$\bigvee_{j\in V} p_{J(\lambda_1)}(i)=\lambda_1 \quad (i\in I=\{i_1,i_2,\cdots,i_r\}\subset U),$$

$$\bigvee_{j\in V} p_{J(\lambda_1)}(i')<\lambda_1 \quad (i'\bar{\in} I).$$

Then, $$Supp[\,p_{J(\lambda_1)}(I)]=J$$

holds. Evidently

$$(p_{J(\lambda_1)}(i_1),\ p_{J(\lambda_1)}(i_2),\cdots p_{J(\lambda_1)}(i_r))_{\lambda_1}=J$$

$x_{i_1}, x_{i_2},\cdots x_{i_k}$ are maximum components of lower solution of $A^{*}\circ X=B$ ($x_{i_1}, x_{i_2},\cdots, x_{i_k}$ was known).

Thus for $\{i_1,i_2,\cdots,i_k\}\subset\{i_1,i_2,\cdots,i_r\}$ $(k\le r)$, then

$$A^{*}_{J(\lambda_1)}\cdot(\{\bigvee_{j\in J(\lambda_1)} p_{J(\lambda_1)}(i_1,i_2,\cdots,i_k)\})^{T}=B_{J(\lambda_1)}.$$

If $\{i_1',i_2',\cdots,i_s'\}\subsetneqq\{i_1,i_2,\cdots,i_k\}$, then

$$A^{*}_{J(\lambda_1)}\cdot(\{\bigvee_{j\in J(\lambda_1)} p_{J(\lambda_1)}(i_1',i_2',\cdots,i_s')\})^{T}\subsetneqq B_{J(\lambda_1)}.$$

thus, $I_t=\{i_1,i_2,\cdots,i_k\}$ is characteristic index set by means of definition 3.

Vice versa.

Corollary 3 Lower solution of λ_1-lumped equation is also $(\{\bigvee_{j\in J} p(i)\})^{T}$, $(i\in I_t)$ where I_t is a characteristic index set.

Definition 10 If lower solution of λ_1-lumped equation is $(\{\bigvee_{j\in J} p_{J(\lambda_1)}(i)\}_{i\in I_t})^{T}$, and there exists $j'\in supp(p(i_1),p(i_2),\cdots,p(i_k))=supp(p(I_t))$ but $j'\bar{\in} J$, let J' be the set of all elements j' in $supp(p(I_t))$ and let $J_1=J\cup J'$. $A^{*}_{J_1(\lambda_1)}\circ X=B_{J_1(\lambda_1)}$ is called the λ_1-lumped extension equation of $A^{*}_{J(\lambda_1)}\circ X=B_{J(\lambda_1)}$

Theorem 7 Lower solution of λ_1-lumped equation is a lower solution of λ_1-lumped extension equation. Vice verse.

Proof. $p^{*}_{J_1(\lambda_1)}\rightleftarrows A^{*}_{J_1(\lambda_1)}$, evidently $\bigvee_{j\in J} p_{J(\lambda_1)}(i_1,i_2,\cdots,i_k)=\bigvee_{j\in J_1} p_{J_1(\lambda_1)}(i_1,i_2,\cdots,i_k)$, because

$$A^{*}_{J(\lambda_1)}\cdot(\bigvee_{j\in J} p_{J(\lambda_1)}(i_1,i_2,\cdots,i_k))^{T}=B_{J(\lambda_1)}$$

then $$A^{*}_{J_1(\lambda_1)}\cdot(\bigvee_{j\in J_1} p_{J_1(\lambda_1)}(i_1,i_2,\cdots i_k))^{T}=B_{J_1(\lambda_1)}.$$

Therefore $I_t=\{i_1,i_2,\cdots,i_k\}$ is characteristic index set.

The theorem 7 presents an important fact: If we seek out lower solution of λ_1-lumped equation, then this solution satisfies voluntarily for λ_1-lumped extension equation. Therefore after determining characteristic Fuzzy set for λ_1 in p, we discard the intersection "$\cap$" of this characteristic Fuzzy set and non-characteristic Fuzzy set from $p(U\setminus I_t)$, i.e. $p(U\setminus I_t)\setminus p(I_t)\cap p(U\setminus I_t)$, afterward operate $\bigvee_{j\in J} p_{J(\lambda_1)}(i_1,i_2,\cdots,i_k)$.

Definition 11 If solution of $A^{*}\circ X=B$ exists, thus we discard λ_1-lumped extension equation from $A^{*}\circ X=B$, therefore the remaining part of $A^{*}\circ X=B$ is called λ_1-surplus lumped equation.

Theorem 8 The second-class component are $x_{i_{K_1}+1}, x_{i_{K_1}+2}, \cdots, x_{i_{K_1}+K_2}$ iff $I_t^{(1)}=\{i_{K_1}+1, i_{K_1}+2, \cdots, i_{K_1}+K_2\}$ is characteristic index set which is determined by the maximum level λ_2' of λ_1-surplus lumped equation (it is called second-class characteristic index set)

Proof. The same as that of theorem 6.

Generally, we have t-th-class component and t-th-class characteristic index set, then

Theorem 9 X_0 is lower solution of $A^{*}\circ X=B$ iff t-th-class components of X_0 correspond to t-th-class characteristic index set.

The process of determining each class component of $A^{*}\circ X=B$ according to each level λ_t of $A^{*}\circ X=B$, one by one, is called the characteristic sum decomposition, denoted by

$$\lambda_1 \dotplus \lambda_2' \dotplus \cdots \dotplus \lambda_{t'}' \qquad (t'\geq 1)$$

Remark. because λ-lumped extension equation is grown in $A^{*}\circ X=B$, then, the number of $\lambda_{t'}$ $(t'\geq 1)$ in characteristic sum is not equal to that of λ_t in $A^{*}\circ X=B$.

To the characteristic sum decomposition, we have

Theorem 10 There is a solution of $A^{*}\circ X=B$, $E, E^{(1)}, E^{(2)}, \cdots, E^{(l)}$ are fuzzy reflexivable permutation which are determined by $l+1$-class characteristic index set one by one, let $E\hat{\gamma}p=p^{(1)}$, $E^{(1)}\hat{\gamma}p^{(1)}=p^{(2)}, \cdots$, then

1° $p \supset E\hat{\gamma}p \supset E^{(1)}\hat{\gamma}p^{(1)} \supset \cdots \supset p^{(l)}\hat{\gamma}E^{(l)}=p^{(l)}\neq\phi$.

2° $(\bigvee_{j\in V} E^{(l)}\hat{\gamma}p^{(l)})^{T}$ is lower solution of $A^{*}\circ X=B$

Proof We can prove 1° according to definition 4 and 5

2° First we prove $A^{*}\circ(\bigvee_{j\in V} E\hat{\gamma}p)^{T}=B$.

$$\begin{aligned}&\bigvee_{i\in U}[a_{ji}^{*}\wedge((\bigvee_{\substack{j\in V\\ i\in I_t}} e_{ij}\gamma r_{ij})\cup(\bigvee_{\substack{j\in V\\ i\bar{\in} I_t}} e_{ij}\hat{\gamma}r_{ij}))]\\ =&\bigvee_{i\in U}[a_{ji}^{*}\wedge((\bigvee_{\substack{j\in V\\ i\in I_t}}\lambda_1\gamma r_{ij})\vee(\bigvee_{\substack{j\in V\\ i\bar{\in} I_t}}\lambda_1\gamma r_{ij}))]\\ =&\bigvee_{i\in U}[a_{ji}^{*}\wedge(\lambda_1\vee r_{ij}')]\end{aligned}$$

According to definition 5 we have $r_{ij}'<\lambda$; and $a_{ji}^{*}\in\{0,\lambda_1\}$, according to theorem 1 and corollary 1, there, $supp(p^{-1}(j))\neq\phi$, then $\forall j$, there exists at least one element in U, say i_0, such that $a_{ji_0}^{*}=b_j$, $b_j=\lambda_1$. Then

$$\bigvee_{i\in U}[a_{ji}^{*}\wedge(\lambda_1\vee r_{ij}')]=\bigvee_{i\in U}[a_{ji}^{*}\wedge\lambda_1]=\bigvee_{i\in U}[a_{ji}^{*}\wedge b_j]=a_{ji_0}^{*}\wedge b_j=b_j.$$

Similarly. $A^{*}\circ(\bigvee_{j\in V}E^{(t)}\hat{\gamma}p^{(t)})^{T}=B \quad (t=1,2,\cdots,l)$

therefore $A^{*}\circ(\bigvee_{j\in V}p)^{T}=A^{*}\circ(\bigvee_{j\in V}E\gamma p)^{T}=\cdots=A^{*}\circ(\bigvee_{j\in V}E^{(l)}\hat{\gamma}p^{(l)})^{T}$

$$=A^{*}\circ(\bigvee_{j\in V}p^{(l)})^{T}=B.$$

also because 1° $p\supset E\hat{\gamma}p\supset\cdots\supset E^{(l)}\hat{\gamma}p^{(l)}=p^{(l)}\neq\phi$

thus $\bigvee_{j\in V}p\geqslant\bigvee_{j\in V}(E\hat{\gamma}p)\geqslant\cdots\geqslant\bigvee_{j\in V}p^{(l)}\neq(0,0,\cdots,0)$.

Then $(\bigvee_{j\in V} p^{(\ell)})^T$ is lower solution of $A^*\circ X=B$.

$p, p^{(1)}, p^{(2)}, \cdots, p^{(l)}$ are obtained by the method provide by means of theorem 10, therefore lower solution is obtained

We illustrate the method using which the lower solution is obtained. If this method is tabulated, it is convenience to find all lower solutions without repretion. by same example 2

First, the iso-solution matrix corresponde to p, i.e. $A^*\to p$.

Next, $\lambda_1=0.8$ and characteristic Fuzzy set $\{R_1 \cup R_2\}$ are determined. 0.8 which is supremum of $\{R_1\cup R_2\}$ is marked down in first row of table. For the functions of $\hat{\gamma}$. We discard $0.8/2$, $0.8/3$, $0.3/9$, $0.4/7$ and $0.5/6$ from $p(U\setminus I_1)$, i.e. $p(3)$, $p(4)$, $p(5)$ and $p(6)$. and mark down these by "x" in first row of table. This process corresponds to the operation $E\hat{\gamma}p$. the selection of first-class component is completed.

The second row, we thake $\lambda_2=0.7$ and place it into second row of the table (0.7 is the supremum of characteristic Fuzzy set $\{R_4\}$). $0.7/4$ is discarded from R_1 and and marked by "x". This process correspond to the operation $E^{(1)}\hat{\gamma}p^{(1)}$. then the selection of second-class component is completed.

Take λ_3, λ_4 be another number, for the same way. we can select third-class and fourth-class component.

The fifth row is sum up first to fourth row. the sixth to fourteenth row are obtained similarly, all are images of $p^{(\ell)}$, i.e. supremum of image of p. There are corresponding lower solutions of $p^{(\ell)}$ image in the right side of the table, i.g. for fifth row,

$$\bigvee_{j\in V} p^{(\ell)}=(0.8,\ 0.8,\ 0.5,\ 0.7,\ 0.4,\ 0)=X_1^T.$$

$X_2=(0.8,\ 0.8,\ 0.7,\ 0,\ 0.4,\ 0)^T$, $X_3=(0.8,\ 0.8,\ 0.7,\ 0,\ 0,\ 0.4)^T$ and so on. (table 1)

In several lower solutions, all first component with same place look as if "tree-root", minimal component look as if "tree-tip", other look as if "tree-joint". Then lower solution tree consist of all lower solution (fig. 2). The decimal in the fig. 2 represents the component of lower solution, and integer in the parentheses represents inverse image, i.e. the place of above lower solution component.

The three lower solution trees are obtained from the above ten lower solutions. The supremum of each-class characteristic Fuzzy set is "tree-root", "tree-joint" and "tree-tip" separately. The branch of "lower solution tree" is a lower solution, for example, the lower solution $(0.8,\ 0.8,\ 0.7,\ 0,\ 0.4,\ 0)^T$ is represented by $\underset{(1,2)}{0.8}\to\underset{(3)}{0.7}\to\underset{(5)}{0.4}$.

Clearly, lower solution tree is not equivalent class.

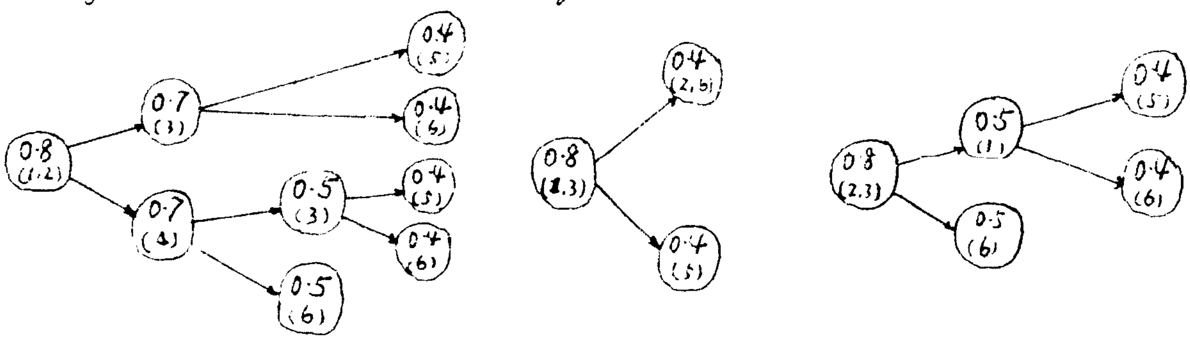

TABLE 1

	$R_i(j)$ Fuzzy Permutation / Characteristic Sum	1: 0.8/1	1: 0.8/1	1: 0.5/6	1: 0.3/9	2: 0.8/1	2: 0.8/3	2: 0.4/7	3: 0.8/2	3: 0.8/3	3: 0.7/4	3: 0.5/5	4: 0.7/4	4: 0.3/9	5: 0.4/7	5: 0.4/8	6: 0.5/5	6: 0.5/6	6: 0.4/8	6: 0.3/9	Lower solution of corresponding Characteristic Sum
1	$\lambda_1=0.8$	0.8	0.8	x	x	0.8	0.8	x	x	x				x	x			x		x	(0.8, 0.8, x, x, x, x)
2	$\lambda_1\dot{+}\lambda_2=0.8\dot{+}0.7$										x		0.7								(0.8, 0.8, x, 0.7, x, x)
3	$\lambda_1\dot{+}\lambda_2\dot{+}\lambda_3=0.8\dot{+}0.7\dot{+}0.5$											0.5					x				(0.8, 0.8, 0.5, 0.7, x, x)
4	$\lambda_1\dot{+}\lambda_2\dot{+}\lambda_3\dot{+}\lambda_4=0.8\dot{+}0.7\dot{+}0.5\dot{+}0.4$															0.4			x		(0.8, 0.8, 0.5, 0.7, 0, 0)
5	$0.8\dot{+}0.7\dot{+}0.5\dot{+}0.4$	0.8	0.8	0	0	0.8	0.8	0	0	0	0	0.5	0.7	0	0	0.4	0	0	0	0	(0.8, 0.8, 0.5, 0.7, 0.4, 0)
6	$0.8\dot{+}0.7\dot{+}0.4$	0.8	0.8	0	0	0.8	0.8	0	0	0	0.7	0	0	0	0	0.4	0	0	0	0	(0.8, 0.8, 0.7, 0, 0.4, 0)
7	$0.8\dot{+}0.7\dot{+}0.4$	0.8	0.8	0	0	0.8	0.8	0	0	0	0.7	0	0	0	0	0	0	0	0.4	0	(0.8, 0.8, 0.7, 0, 0, 0.4)
8	$0.8\dot{+}0.7\dot{+}0.5$	0.8	0.8	0	0	0.8	0.8	0	0	0	0	0	0.7	0	0	0	0.5	0	0	0	(0.8, 0.8, 0, 0.7, 0, 0.5)
9	$0.8\dot{+}0.7\dot{+}0.5\dot{+}0.4$	0.8	0.8	0	0	0.8	0.8	0	0	0	0	0.5	0.7	0	0	0	0	0	0.4	0	(0.8, 0.8, 0.5, 0.7, 0, 0.4)
10	$0.8\dot{+}0.4$	0.8	0.8	0	0	0	0	0.4	0.8	0.8	0	0	0	0	0	0	0	0	0.4	0	(0.8, 0.4, 0.8, 0, 0, 0.4)
11	$0.8\dot{+}0.4$	0.8	0.8	0	0	0	0	0	0.8	0.8	0	0	0	0	0.4	0.4	0	0	0	0	(0.8, 0, 0.8, 0, 0.4, 0)
12	$0.8\dot{+}0.5\dot{+}0.4$	0	0	0.5	0	0.8	0.8	0	0.8	0.8	0	0	0	0	0	0.4	0	0	0	0	(0.5, 0.8, 0.8, 0, 0.4, 0)
13	$0.8\dot{+}0.5\dot{+}0.4$	0	0	0.5	0	0.8	0.8	0	0.8	0.8	0	0	0	0	0	0	0	0	0.4	0	(0.5, 0.8, 0.8, 0, 0, 0.4)
14	$0.8\dot{+}0.5$	0	0	0	0	0.8	0.8	0	0.8	0.8	0	0	0	0	0	0	0	0.5	0	0	(0, 0.8, 0.8, 0, 0, 0.5)

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