# Existence and Uniqueness Theorem of Standard Basis of Fuzzy Module

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## Abstract

In this paper, a new concept on fuzzy module is presented, and the existence and uniqueness theorem of standard basis of fuzzy module is given, i.e. finitely generated sequential fuzzy module has a sole standard basis under minimal condition.

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We discuss the linear spaces based on fuzzy algebra $[0, 1]$.

Definition 1. Let V be a nonempty set, R be a fuzzy algebra $[0, 1]$ with the operations $a + b = \sup\{a, b\}$, $ab = \inf\{a, b\}$, if V is a linear space over the fuzzy algebra R, it is called a fuzzy module.

Definition 2. A subset W of fuzzy module V is a fuzzy submodule iff $ku + lv \in W$, $(\forall u, v \in W, \forall k, l \in R)$

Example. The set of all solution vectors of fuzzy relation equation $AX = B$ (A, B are fuzzy matrices, X is a fuzzy vector) is a solution submodule.

Proposition 1. Let S be a nonempty set of fuzzy module V. The intersection of all fuzzy submodules of V which contains S is a fuzzy submodule. It's called a span of S, and is denoted by $\langle S \rangle$.

Definition 3. A basis for fuzzy module V is a minimal spanning set for the fuzzy module.

Definition 4. A fuzzy module W is independent iff U is not a linear combination of elements of $W \setminus \{U\}$ for any $U \in W$. If there is a $U \in W$ which is a linear combination of elements of $W \setminus \{U\}$ it is said to be dependent.

Proposition 2. Every basis of fuzzy module V is independent.

Proof let B be a dependent basis in V, there exist $b \in B$, which is a linear combination of elements of $B\setminus\{b\}$. Since $B\setminus\{b\} \subsetneq$ and $B\setminus\{b\}$ is a set of generators of V. This is a contradiction. This completes the proof.

A fuzzy module V is said to be finite spanning if set of generators of V is a finite set.

Definition 5. A fuzzy module V is called a sequential fuzzy module if and only if V is a sequential set, such that

$$ru \leq u \qquad (\forall r \in R,\ u \in V)$$

$$u_1 + u_2 \geq \sup\{u_1, u_2\} \qquad (u_1, u_2 \in V)$$

Definition 6. A basis C of a fuzzy module V is called a standard basis iff whenever $c_i = \sum r_{ij} c_j$ for $c_i, c_j \in C$ then $c_i = r_{ii} c_i$

Definition 7. A sequential fuzzy module V is minimal conditional iff a chain of any subset of V.

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$

The process will end in a finite steps, i.e. there exists a natural number N, such that

$$a_N = a_{N+1} = \cdots = a_{N+p} = \cdots$$

Proposition 3. If sequencial fuzzy module V is minimal conditional. Then $V \times V \times \cdots \times V$ is minimal conditional too.

Proof A chain of any subset of $V \times V \times \cdots \times V$,

$$(a_1^{(1)}, a_1^{(2)}, \cdots, a_1^{(n)}) \geq (a_2^{(1)}, a_2^{(2)}, \cdots, a_2^{(n)}) \geq \cdots \geq (a_n^{(1)}, \cdots, a_n^{(n)})$$

hence

$$a_1^{(i)} \geq a_2^{(i)} \geq \cdots \geq a_n^{(i)} \geq \cdots \qquad i = 1, 2, \cdots, n,$$

Since V is minimal conditioned, then for any $i$, there exists a natural number N(i), such that

$$a_{N(i)}^{(i)} = a_{N+1(i)}^{(i)} = \cdots = a_{N+p(i)}^{(i)} = \cdots$$

Let N = max(N(1), N(2), ..., N(n). Then

$$(a_N^{(1)}, a_N^{(2)}, \cdots, a_N^{(n)}) = (a_{N+1}^{(1)}, \cdots, a_{N+1}^{(n)}) = \cdots$$

The proposition is proved.

Proposition 4. Let $C_1, C_2, \ldots, C_k, \ldots, C_n$ be a standard basis of sequential fuzzy module V, and $C_k = \sum_{j=1}^{n} a_j$, $a_j \in V$, Then $C_k = a_j$ ;

Proof $$a_j = \sum_{i=1}^{n} a_{ji} C_i$$

Therefore
$$C_i = \sum_{j=1}^{n} a_j$$
$$= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ji} C_i \right)$$
$$= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ji} \right) C_i$$

From the definition of the standard basis,

$$\left( \sum_{j=1}^{n} a_{jk} \right) C_k = C_k$$

Because $\sum_{j=1}^{n} a_{jk} = \bigvee_j a_{jk}$ Then $a_{jk} C_k = C_k$ fo some $j$

Therefore

$$a_j = \sum_{i=1}^{n} a_{ji} C_i = C_k + \sum_{i \neq k} a_{ji} C_i ,$$

by definition 5, implies that

$$a_j \geq C_k$$

Conversely, from $C_K = \sum_{j=1}^{n} a_j$ we know that $C_K \geq a_j$

Therefore $C_K = a_j$ . This completes the proof.

Theorem. (Existence and uniqueness theorem of standard basis of fuzzy module).

Finitely generated sequential fuzzy modules has a sole standard base under minimal condition.

Proof $1^{\circ}$ Existence.

Let V be a finitely generated sequential fuzzy modules under minimal condition, and C be any finite basis, $C = \{C_1, C_2, \ldots, C_n\}$

Suppose C is not standard.

Then $C_i = \sum_{j=1}^{n} a_{ij} C_j$ for some $C_i \in C$, $C_i \neq a_{ii} C_i$ by definition 5,

$a_{ii} C_i \leq C_i$, We have $a_{ii} C_i < C_i$, Let $C^{(1)}$ be the n tuples of clements of V obtained from C by replacing $C_i$ by $a_{ii} C_i$ .

Then $C^{(1)} = \{ C_1^{(1)}, C_2^{(1)}, \cdots, C_n^{(1)} \}$

$C^{(1)}$ is a basis of V too.

Let $C = (C_1, C_2, \cdots, C_n) \in V \times V \times \cdots \times V$.

$C^{(1)} = (C_1^{(1)}, C_2^{(1)}, \cdots, C_n^{(1)}) \in V \times V \times \cdots \times V$

Straightforward $C > C^{(1)}$

If $C^{(1)}$ is not standard either, then $C_i^{(1)} = \sum_{j=1}^{n} a_{ij}^{(1)} C_j^{(1)}$

for some $C_i \in C$,

$$C_i^{(1)} > a_{ii}^{(1)} C_i^{(1)}$$

Let $C_i^{(2)} = a_{ii}^{(1)} C_i^{(1)}$, we have

$$C^{(2)} = ( C_1^{(2)}, C_2^{(2)}, \cdots, C_n^{(2)} )$$

go on in this manner, we obtain a descending chain,

C C

$$C > C^{(1)} > C^{(2)} > \cdots\cdots$$

But since $V \times V \times \cdots \times V$ is minical conditioned, then there exists a natural number N, such that

$$C^{(N)} = C^{(N+1)} = \cdots = C^{(N+P)} = \cdots \qquad \forall P$$

This proves that for any finite basis, there exists a standard basis.

2° Uniqueness.

Let $C = \{C_1, C_2, \ldots, C_n\}$ and $C' = \{C'_1, C'_2, \ldots C'_n\}$

be two standard basis of the fuzzy module V.

Then $\quad C_i = \sum_{j=1}^{n} \gamma_{ij} C'_j \qquad i = 1, 2, \cdots, n$

By proposition 4, $\exists j$, such that

$$C_i = \gamma_{ij} C'_j$$

since $\quad C'_j = \sum_{K=1}^{n} \ell_{iK} C_K, \quad \exists K,$ such that

$$C'_j = \ell_{iK} C_K$$

Therefore $\quad C_i = \gamma_{ij} \ell_{iK} C_K$

We conclude that, $\quad C_i = C_K$

If not, we can express $C_i$ in term of sums of multiples of elements of $C \setminus \{C_i\}$ This is a contradiction.

i.e. $\quad C_i = \gamma_{ij} C'_j, \quad C'_j = \ell_{iK} C_i$

Then $\quad C_i \leq C'_j, \quad C'_j \leq C_i$

Therefore $\quad C_i = C'_j$

If $C_i = C_k'$ then $C_k' = C_j'$, This is a contradiction.
By rearrangeing the order, we have

$$C_i = C_i' \qquad i = 1, 2, \cdots, n$$

Corollary. Any two basis for finitely generated sequential fuzzy modules under minimal condition have the same cardinality.

References


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