CALCULUS OF FUZZY SETS I*)

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Summary. We are given a general view on dependence between properties of poset L and the family of L-fuzzy sets. It is an exemplification of a previous Goguen's idea (cf.[10]).

1. L-fuzzy sets. Let $X \neq \emptyset$ denotes an arbitrary set and let $(L, \leqslant)$ be a bounded poset (partially ordered set) with bounds denoted by 0 and 1.

Definition 1 (Zadeh [21], Goguen [10]). An L-fuzzy set (briefly L-set or fuzzy set) in X is a mapping $A: X \to L$. The family of all L-sets in X is denoted by $L(X)$.

For any crisp set $M \subset X$, its characteristic function 1_M,

$$(1) \qquad 1_M(x) = \begin{cases} 1 & \text{for } x \in M, \\ 0 & \text{for } x \in X \setminus M \end{cases}$$

is a special case of L-fuzzy set. $1_\emptyset$ is also denoted by 0_X and generally for $r \in L$ we put

$$(2) \qquad r_M(x) = \begin{cases} r & \text{for } x \in M, \\ 0 & \text{for } x \in X \setminus M, \end{cases}$$

so $r_M \in L(X)$.

$L(X)$ is partially ordered as a family of mappings taking values in poset, namely

$$(3) \qquad A \leqslant B \Leftrightarrow (A(x) \leqslant B(x) \ \text{ for } x \in X)$$

and it has bounds 0_X and 1_X (for the characteristic functions (1) the relation (3) coincides with the inclusion of sets).

The following notions are useful for illustration of fuzzy concepts:

Definition 2 (Kloeden [13]). The set

$$(4) \qquad G(A) = \{(x,r) \in X \times L \mid r \leqslant A(x) \}$$

is called an endograph of L-fuzzy set A.

Definition 3 (Zadeh [21], Weiss [20]). The crisp sets

$$(5) \quad N_r(A) = \{x \in X \mid A(x) \geqslant r \}, \quad M_r(A) = \{x \in X \mid A(x) > r \} \quad \text{for } r \in L$$

are called cuts (r-cuts)**) and strong cuts of L-fuzzy set A, respectively.

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*) It is a part of [8], Chap.2.

**) Term is connected with geometrical interpretation of endograph alike the term "level sets" used for (5) in literature (cf.[23]). Another cuts (fuzzy levels) are considered in [16].

$M_0(A)$ is also called support of A and it is denoted by S(A),

(6) $$S(A) = \{x \in X \mid A(x) > 0\} .$$

It is obvious that for $A \in L(X)$ we get

$$N_0(A) = X, \quad M_1(A) = \emptyset ,$$

$$M_r(A) \subset N_r(A) \quad \text{for} \quad r \in L,$$

$$r \leqslant s \Rightarrow (N_s(A) \subset N_r(A),\ M_s(A) \subset M_r(A)) \quad \text{for} \quad r, s \in L$$

and for $A, B \in L(X)$ we have

(7) $$A \leqslant B \Leftrightarrow G(A) \subset G(B) ,$$

(8) $$A \leqslant B \Leftrightarrow N_r(A) \subset N_r(B) \quad \text{for} \quad r \in L ,$$

(9) $$A \leqslant B \Rightarrow M_r(A) \subset M_r(B) \quad \text{for} \quad r \in L .$$

The implication inverse to (9) is obtained only for linear L and then

(10) $$A = B \Leftrightarrow M_r(A) = M_r(B) \quad \text{for} \quad r \in L .$$

Any L-set is uniquely determined by the family of all its cuts:

Theorem 1 (Resolution identity*). For any $A \in L(X)$ there exists the supremum

(11) $$\bigvee_{r \in L} r_{N_r(A)} = A .$$

Proof. Let $x \in X$. Putting

(12) $$A_r = r_{N_r(A)} \quad \text{for} \quad r \in L ,$$

we prove that

(13) $$\bigvee_{r \in L} A_r(x) = A(x) ,$$

which is equivalent to (11). By (2) and (5) we get

$$A_r(x) \leqslant A(x) \quad \text{for} \quad r \in L$$

and therefore set $\{A_r(x)\}_{r \in L}$ has upper bound A(x). But for $r = A(x) \in L$ we get $A_r(x) = A(x)$ which implies (13).

From the algebraic point of view L(X) is a direct product of posets, $L(X) = L^X$. This gives an useful method for introduction of different structures in L(X). After an argumentation of Goguen [11] we can put

Theorem 2. Any property can be extended form L to L(X) iff it is conserved by direct product operation.

Particular cases of this theorem were cosidered in [3]-[6] and [14]. We consider its application for the lattice algebra and for ordered groupoids.

*) For $L = [0,1]$ the resolution identity was proved by Zadeh [22].

2. Lattices of L-fuzzy sets. Properties presented here are related to these in the algebra of sets. After Theorem 2 we get (cf. [12], § 46)

Theorem 3. L(X) is a bounded, complete, distributive, infinitely distributive, completely distributive, Brouwerian, De Morgan or Boolean lattice iff L has respective property. Moreover, operations from L to L(X) are extended pointwise and operations from L(X) to L are projected by using constant fuzzy sets r_X for $r \in L$ (cf.(2)).

For particular L we get some consequences of Theorem 3.

Corollary 1. If $(L,\vee,\wedge)$ is a bounded lattice then $(L(X),\vee,\wedge)$ is also a bounded lattice, where lattice operations for $A, B \in L(X)$ have the form

(14) $(A\vee B)(x) = A(x)\vee B(x)$, $(A\wedge B)(x) = A(x)\wedge B(x)$ for $x \in X$.

In particular for $A, B, C, D \in L(X)$ we get (cf. [2], Chap.I)

$$A\wedge B \leqslant A \leqslant A\vee B ,$$

$$A \leqslant B \iff A\wedge B = A \iff A\vee B = B ,$$

$$(C \leqslant A,\ C \leqslant B) \iff C \leqslant A\wedge B ,$$

$$(A \leqslant D,\ B \leqslant D) \iff A\vee B \leqslant D ,$$

$$A\vee A = A , \qquad A\wedge A = A ,$$

$$A\vee B = B\vee A , \qquad A\wedge B = B\wedge A ,$$

$$A\vee (B\vee C) = (A\vee B)\vee C , \qquad A\wedge (B\wedge C) = (A\wedge B)\wedge C ,$$

$$A\vee (A\wedge B) = A, \qquad A\wedge (A\vee B) = A ,$$

$$A \leqslant B \Rightarrow (A\vee C \leqslant B\vee C,\ A\wedge C \leqslant B\wedge C) ,$$

$$(A\wedge B)\vee (C\wedge D) \leqslant (A\vee C)\wedge (B\vee D) ,$$

$$(A\wedge B)\vee C \leqslant (A\vee C)\wedge (B\vee C) , \quad (A\wedge C)\vee (B\wedge C) \leqslant (A\vee B)\wedge C ,$$

$$(A\wedge B)\vee (B\wedge C)\vee (C\wedge A) \leqslant (A\vee B)\wedge (B\vee C)\wedge (C\vee A) ,$$

$$A\wedge 0_X = 0_X , \qquad A\vee 0_X = A ,$$

$$A\wedge 1_X = A , \qquad A\vee 1_X = 1_X .$$

The lattice product (14) can be characterized by means of endographs and cuts

(15) $C = A\wedge B \iff G(C) = G(A)\cap G(B)$,

(16) $C = A\wedge B \iff (N_r(C) = N_r(A)\cap N_r(B)$ for $r \in L)$.

For the lattice sum we only have

$$G(A)\cup G(B) \subset G(A\vee B) ,$$

$$N_r(A)\cup N_r(B) \subset N_r(A\vee B) \quad \text{for } r \in L ,$$

but if lattice L is linear then we get a characterization

(17) $C = A \vee B \iff G(C) = G(A) \vee G(B)$,

(18) $C = A \vee B \iff (N_r(C) = N_r(A) \vee N_r(B) \text{ for } r \in L)$.

The strong cuts are less useful for such characterization. In general we get

$$M_r(A \wedge B) \subset M_r(A) \cap M_r(B), \quad M_r(A) \vee M_r(B) \subset M_r(A \vee B) \quad \text{for } r \in L$$

however for linear L we have properties similar to (16) and (18).

Corollary 2. If L is a distributive lattice then L(X) is also a distributive lattice and for $A, B, C \in L(X)$ we get (cf. [2], Chap.II)

$$(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C) \leqslant A \vee (B \wedge C) \ ,$$

$$A \wedge (B \vee C) \leqslant (A \vee C) \wedge (B \vee C) = (A \wedge B) \vee C \ ,$$

$$(A \wedge B) \vee (B \wedge C) \vee (C \wedge A) = (A \vee B) \wedge (B \vee C) \wedge (C \vee A) \ ,$$

$$A = B \iff (A \vee C = B \vee C, \ A \wedge C = B \wedge C) \ ,$$

$$A \subset B \iff (A \vee C \subset B \vee C, \ A \wedge C \subset B \wedge C) \ .$$

Corollary 3. If L is a complete lattice, then L(X) is also a complete lattice and for any $A_t \in L(X)$, $t \in T$ we have

(19) $(\bigvee_{t \in T} A_t)(x) = \bigvee_{t \in T} A_t(x) \ , \quad (\bigwedge_{t \in T} A_t)(x) = \bigwedge_{t \in T} A_t(x) \quad \text{for } x \in X$.

If $A, A_t, B_t, C_{st}, D_s \in L(X)$ for $s \in S$, $t \in T$ then we also have (cf. [2], Chap.V)

$$\bigvee_{s \in S} (\bigvee_{t \in T} C_{st}) = \bigvee_{t \in T} (\bigvee_{s \in S} C_{st}) \ , \quad \bigwedge_{s \in S} (\bigwedge_{t \in T} C_{st}) = \bigwedge_{t \in T} (\bigwedge_{s \in S} C_{st}) \ ,$$

$$\bigvee_{s \in S} (\bigwedge_{t \in T} C_{st}) \leqslant \bigwedge_{t \in T} (\bigvee_{s \in S} C_{st}) \ , \quad \bigvee_{s \in S} \bigvee_{t \in T} (A_t \wedge D_s) \leqslant (\bigvee_{t \in T} A_t) \wedge (\bigvee_{s \in S} D_s) \ ,$$

$$(\bigwedge_{t \in T} A_t) \vee (\bigwedge_{s \in S} D_s) \leqslant \bigwedge_{t \in T} \bigwedge_{s \in S} (A_t \vee D_s) \ ,$$

$$(\bigwedge_{t \in T} A_t) \vee (\bigwedge_{t \in T} B_t) \leqslant \bigwedge_{t \in T} (A_t \vee B_t), \quad \bigvee_{t \in T} (A_t \wedge B_t) \leqslant (\bigvee_{t \in T} A_t) \wedge (\bigvee_{t \in T} B_t) \ ,$$

$$A \vee (\bigwedge_{t \in T} B_t) \leqslant \bigwedge_{t \in T} (A \vee B_t) \ , \quad \bigvee_{t \in T} (A \wedge B_t) \leqslant A \wedge (\bigvee_{t \in T} B_t) \ ,$$

$$(A_t \leqslant B_t \text{ for } t \in T) \Rightarrow (\bigvee_{t \in T} A_t \leqslant \bigvee_{t \in T} B_t \ , \ \bigwedge_{t \in T} A_t \leqslant \bigwedge_{t \in T} B_t) \ .$$

Using endographs and cuts we get

(20) $C = \bigwedge_{t \in T} A_t \iff G(C) = \bigcap_{t \in T} G(A_t)$.

(21) $$\bigcup_{t\in T} G(A_t) \subset G(\bigvee_{t\in T} A_t) ,$$

... for $r \in L$) ,