SOME REMARKS ON ROUGH AND FUZZY SETS

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SUMMARY. Rough sets have been introduced by Z.Pawlak. They can be defined in the language of the fuzzy set theory by means of three-valued membership functions. In this paper we show that basic operations on rough sets (i.e. union and intersection) can be expressed as some special operations on their membership functions.

1. ROUGH SETS

Rough sets were introduced by Z.Pawlak in 1981 (see /4/-/7/). We are going to outline that theory.

Let U denote an ordinary set called universe of discourse. Let R denote an equivalence relation (defined on U) which will be considered to be an indiscernibility relation. Then the pair A=(U,R) is called approximation space. All the further considerations will be related to the fixed approximation space A.

The equivalence classes of R (and the empty set $\emptyset$, too) will be called elementary sets in A. The equivalence class containing $x\in U$ is as usually denoted by $[x]_R$. Any union of elementary sets is instead called composed set in A. The family of all the composed sets in A will be denoted by Com(A).

Let X denote an ordinary subset of U, and let

$$\underline{X}:=\{x\in U:\ [x]_R\subset X\},\quad \overline{X}:=\{x\in U:\ [x]_R\cap X\neq\emptyset\},$$

where := stands for "equals by the definition". Then $\underline{X}$ ($\overline{X}$, resp.) is called best lower (best upper, resp.) approximation of X in A. So, $\underline{X}$ is the greatest composed set included in X, and $\overline{X}$ is the least composed set including X.

One can easily check (see /6/,/7/) that the pair $T_A=(U,Com(A))$ constitutes a topological space, where Com(A) is the family of all open sets in T_A. Elementary sets are a base for T_A. Moreover, Com(A) is both the family of all open and closed sets in T_A.

From the definition of $\underline{X}$ and $\overline{X}$ it follows that $\underline{X}$ and $\overline{X}$ are interior and closure of X in T_A. Thus the following properties hold:

(A1) $\underline{X}\subset X\subset\overline{X}$,

(A2) $\underline{U}=\overline{U}=U$, $\underline{\emptyset}=\overline{\emptyset}=\emptyset$,

(A3) $\overline{X\cup Y}=\overline{X}\cup\overline{Y}$,

(A4) $\underline{X}\cup\underline{Y}\subset\underline{X\cup Y}$,

(A5) $\overline{X\cap Y}\subset\overline{X}\cap\overline{Y}$,

(A6) $\underline{X\cap Y}=\underline{X}\cap\underline{Y}$,

(A7) $\underline{X^c}=(\overline{X})^c$, $\overline{X^c}=(\underline{X})^c$, where X^c denotes the complement of X.

Moreover, $X\subset Y$ implies $\underline{X}\subset\underline{Y}$ and $\overline{X}\subset\overline{Y}$; $X=\underline{X}=\overline{X}$ iff $X\in Com(A)$.

Other interesting properties of the best approximations are listed in /5/,/6/. They are , in essence, various counterparts of the De Morgan's laws and of the classical rules $X\cup X^c=U$, $X\cap X^c=\emptyset$, expressed by means of $\underline{X}$ and $\overline{X}$.

Using the notion of the best(lower and upper) approximation one can define two membership predicates $\underline{\in}$ and $\overline{\in}$, namely:

$x\,\underline{\in}\,X$ iff $x\in\underline{X}$ (strong membership),

$x\,\overline{\in}\,X$ iff $x\in\overline{X}$ (weak membership).

One can say that x surely (possibly, resp.) belongs to X in A if $x\,\underline{\in}\,X$ ($x\,\overline{\in}\,X$, resp.).

Let $X,Y\subset U$. The following three kinds of rough inclusions and equalities can be introduced in the approximation space A (/5/,/6/)

$X\,\underset{\sim}{\subset}\,Y$ iff $\underline{X}\subset\underline{Y}$ (X is a rough lower-subset of Y in A),

$X\,\overset{\sim}{\subset}\,Y$ iff $\overline{X}\subset\overline{Y}$ (X is a rough upper-subset of Y in A),

$X\,\underset{\sim}{\overset{\sim}{\subset}}\,Y$ iff $X\,\underset{\sim}{\subset}\,Y$ and $X\,\overset{\sim}{\subset}\,Y$ (X is a rough subset of Y in A),

and

$X\eqsim Y$ iff $\underline{X}=\underline{Y}$ (X,Y are roughly bottom-equal in A),

$X\simeq Y$ iff $\overline{X}=\overline{Y}$ (X,Y are roughly top-equal in A),

$X\approx Y$ iff $X\eqsim Y$ and $X\simeq Y$ (X,Y are roughly equal in A).

Full list containing different properties of the above-defined rough inclusions and equalities is placed in /5/,/6/.

Now, we are ready to recall the notion of rough set. It is quite obvious that $\eqsim,\simeq,\approx$ are equivalence relations on P(U), where P(U) denotes the power set of U. Equivalence classes of $\approx$

($\approx, \simeq$, resp.) are called rough (lower, upper, resp.) sets. So, a rough (lower, upper, resp.) set is a family of such subsets of U which are "identical" with respect to the indiscernibility relation $\approx$ ($\approx, \simeq$, resp.). Z.Pawlak presents in /4/,/6/ some applications of rough sets and his approximation spaces to various branches of artificial intelligence.

2. ROUGH SETS - SOME LOGICAL AND SET-THEORETIC ASPECTS

The theory of rough sets is an element of the family of set theories related to the three-valued logic. That family comprises, among other, partial set theory and flou sets (see /2/; cf. /10/). Connections between rough and partial sets are described in /9/; however, that research problem has been initiated in /8/.

Although rough sets are defined as equivalence classes of ordinary subsets of U, we cannot wholly use this fact for introducing basic operations on that sets. Namely, it would be very convenient to have $X_1 \overset{\cup}{\cap} Y_1 \approx X_2 \overset{\cup}{\cap} Y_2$ if $X_1 \approx X_2$ and $Y_1 \approx Y_2$. However, the desirable rough equality of unions (intersections) does not hold in general case. Therefore if we like to introduce basic operations on rough sets, we must previously change (i.e. simplify) their definition. Such a simplification is used in /5/,/7/, too.

From now on, by a rough set V we shall mean an imprecisely described set characterized by a pair $[\underline{V}, \overline{V}]$, where $\underline{V}, \overline{V}$ denote (respectively) the best lower and the best upper approximation of V in A=(U,R). We write then $V=[\underline{V}, \overline{V}]$ and assume that the impreciseness of the description of V arises from the lack of complete information about the localization of V rather than fuzzy nature of V. Moreover, let $FrV := \overline{V} - \underline{V}$ (the so-called boundary of V in A).

3. OPERATIONS ON ROUGH SETS AS OPERATIONS ON FUZZY SETS

Let $Y=[\underline{Y}, \overline{Y}]$ and $Z=[\underline{Z}, \overline{Z}]$ denote two rough sets in A. They can be equivalently expressed by means of three-valued membership functions, namely:

$$Y(x)=\begin{cases} 1 & \text{iff } x\in\underline{Y} \\ 0.5 & \text{iff } x\in FrY \\ 0 & \text{iff } x\in(\overline{Y})^c \end{cases}, \qquad Z(x)=\begin{cases} 1 & \text{iff } x\in\underline{Z} \\ 0.5 & \text{iff } x\in FrZ \\ 0 & \text{iff } x\in(\overline{Z})^c \end{cases}.$$

Let $Y \sqcup Z$ and $Y \sqcap Z$ denote union and intersection of Y and Z constructed in accordance with the simplified definition of a rough set, i.e.

$$Y \sqcup Z = [\underline{Y \cup Z}, \overline{Y \cup Z}] , \quad Y \sqcap Z = [\underline{Y \cap Z}, \overline{Y \cap Z}] ,$$

where

$$\underline{Y \cup Z} = \{x \in U : [x]_R \subset Y \cup Z\} , \quad \overline{Y \cup Z} = \{x \in U : [x]_R \cap (Y \cup Z) \neq \emptyset\} ,$$
$$\underline{Y \cap Z} = \{x \in U : [x]_R \subset Y \cap Z\} , \quad \overline{Y \cap Z} = \{x \in U : [x]_R \cap (Y \cap Z) \neq \emptyset\} .$$

Using properties (A3)-(A6) we get

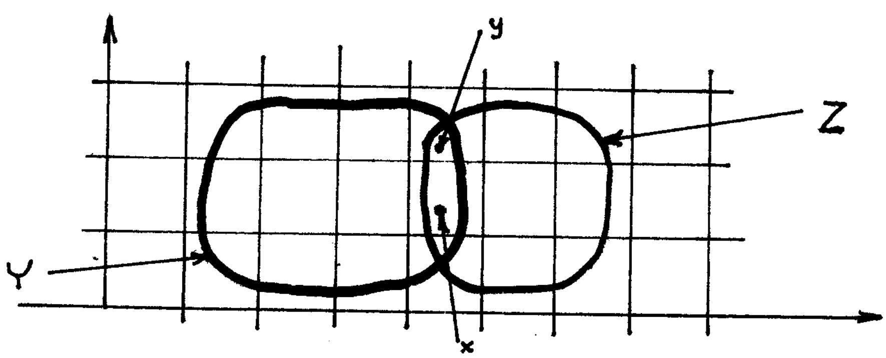
$$Y \sqcup Z = [\underline{Y} \cup \underline{Z}, \overline{Y} \cup \overline{Z}] \text{ and } Y \sqcap Z = [\underline{Y} \cap \underline{Z}, \overline{Y \cap Z}] .$$

Z.Pawlak proved in /7/ that

$$(\&) \quad \begin{aligned} (Y \sqcup Z)(x) &\neq \max(Y(x), Z(x)) \\ (Y \sqcap Z)(x) &\neq \min(Y(x), Z(x)) \end{aligned}$$

and that the equalities in (&) are fulfilled iff $\underline{Y \cup Z} = \underline{Y} \cup \underline{Z}$ and $\overline{Y \cap Z} = \overline{Y} \cap \overline{Z}$, what however does not hold in general case (see (A4), (A5)). Therefore he concludes in /7/ that rough sets are more wide notion than three-valued fuzzy sets, and that basic operations on rough sets cannot be formulated in the language of the fuzzy set theory.

Let us reconsider the problem of operations on rough sets. To this end, it is very instructive to consider the following concrete example. Let $U = \mathbb{R}^2$, and let the equivalence relation R be defined as follows: xRy iff $\text{entier}(x_1) = \text{entier}(x_2)$ and $\text{entier}(y_1) = \text{entier}(y_2)$, where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. Rough sets Y,Z are sketched on the below given picture.



Then $Y(x) = Z(x) = Y(y) = Z(y) = 0.5$ but $(Y \sqcup Z)(x) = 1$ and $(Y \sqcup Z)(y) = 0.5$. Thus if we like to find such operation $\circledcirc$ that

$$(Y \sqcup Z)(x) = Y(x) \circledcirc Z(x) \quad \forall x \in U ,$$

then $\circledcup$ cannot be homogeneously (i.e. identically for all x from U) defined. Really, in the given example we have

$$Y(x)\circledcup Z(x) = 1 \quad (\text{i.e. } 0.5\circledcup 0.5 = 1)$$

and

$$Y(y)\circledcup Z(y) = 0.5 \quad (\text{i.e. } 0.5\circledcup 0.5 = 0.5).$$

Analogous considerations and conclusions can be presented for $Y\sqcap Z$ and for some operation $\circledcap$.

Now we like to show that basic operations on rough sets can be formulated in the language of the fuzzy set theory using heterogeneously defined operations on membership functions. More precisely, we like to express union (intersection, resp.) of two rough sets by means of maximum and bounded-sum (minimum and bounded-product, resp.) of their three-valued membership functions (see /1,3/).

<u>Proposition 1.</u> For any rough sets Y,Z and for every $x\in U$

$$(Y\sqcap Z)(x)=\begin{cases}\max(0,Y(x)+Z(x)-1) & \text{if } Y(x)=Z(x)=0.5 \text{ and } [x]_R\cap(Y\cap Z)=\emptyset,\\ \min(Y(x),Z(x)) & \text{otherwise.}\end{cases}$$

Proof. We have $\overline{Y\cap Z}\subset\overline{Y}\cap\overline{Z}$. So, there exists such a set $F\subset U$ that $\overline{Y\cap Z}=(\overline{Y}\cap\overline{Z})-F$, where $F\cap\overline{Y\cap Z}=\emptyset$ and $F\cup\overline{Y\cap Z}=\overline{Y}\cap\overline{Z}$. Let us note that

$(Y\sqcap Z)(x)=1$ iff $Y(x)=Z(x)=1$,
$(Y\sqcap Z)(x)=0.5$ iff $[(Y(x)=1$ and $Z(x)=0.5)$ or $(Y(x)=0.5$ and $Z(x)=1)$ or $(Y(x)=Z(x)=0.5)]$ and $x\in F^c$,
$(Y\sqcap Z)(x)=0$ iff $Y(x)=0$ or $Z(x)=0$ or $x\in F$.

Moreover $x\in F$ iff $Y(x)=Z(x)=0.5$ and $[x]_R\cap(Y\cap Z)=\emptyset$. Hence

$(Y\sqcap Z)(x)=1$ iff $Y(x)=Z(x)=1$,
$(Y\sqcap Z)(x)=0.5$ iff $(Y(x)=1$ and $Z(x)=0.5)$ or $(Y(x)=0.5$ and $Z(x)=1)$ or $(Y(x)=Z(x)=0.5$ and $[x]_R\cap(Y\cap Z)\neq\emptyset)$,
$(Y\sqcap Z)(x)=0$ iff $Y(x)=0$ or $Z(x)=0$ or $(Y(x)=Z(x)=0.5$ and $[x]_R\cap(Y\cap Z)=\emptyset)$.

The final formula is now quite clear.

<u>Proposition 2.</u> For any rough sets Y,Z and for every $x\in U$

$$(Y\sqcup Z)(x)=\begin{cases}\min(1,Y(x)+Z(x)) & \text{if } Y(x)=Z(x)=0.5 \text{ and } [x]_R\subset Y\cup Z ,\\ \max(Y(x),Z(x)) & \text{otherwise.}\end{cases}$$

Proof. It is analogous to that of Proposition 1.

Remark. Taking into account the definitions of $\overline{Y\cap Z}$ and $\underline{Y\cup Z}$ we can assume that we are always (i.e. for any $x\in U$) able to check whether $[x]_R\cap(Y\cap Z)=\emptyset$ and $[x]_R\subset Y\cup Z$.

Let $\neg Y$ denote the complement of Y constructed in the language of the rough set theory. One can show (see /7/) that $(\neg Y)(x)=1-Y(x)$, i.e. $\neg Y=[(\overline{Y})^c,(\underline{Y})^c]$.

So, the three elementary operations on rough sets (complement, union, intersection) can be expressed in the language of the fuzzy set theory. Therefore, from the viewpoint of definition and basic operations, rough sets may be considered as a particular case of fuzzy sets.

Acknowledgements

This paper has been presented on the Seminar on Interval and Fuzzy Mathematics (Technical University of Poznań). The author wishes to thank to Professor Jerzy Albrycht as well as to many members of the seminar for remarks and discussion.

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