A STRUCTURAL REPRESENTATION OF THE TOTALLY FUZZY SETS.

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Many works have been done about the fuzzy sets, particularly about the fuzzy sets with a relation of indiscernability, called "the totally fuzzy sets". Applying the definition of the category of the totally fuzzy sets exposed by PONASSE (1), we will give a way of showing in order to represent the structure of a totally fuzzy set.

Thereafter, J is always designed by a finite sequence :

$$J = (r_0, r, \dots, r_n) \text{ with } 0 = r_0 < \dots < r_n = 1.$$

We recall the essential definition of JTF : the category of the J-totally fuzzy sets :

.The objects of JTF are the J-totally fuzzy sets, i-e the triplets $\underline{A} = (A, \alpha, \sigma)$ where A is a set, $\sigma : A \times A \longrightarrow J$, $\alpha : A \longrightarrow J$ are two functions verifying :

$$\sigma(a,a) = \alpha(a), \quad \sigma(a,b) = \sigma(b,a), \quad \sigma(a,b) \wedge \sigma(b,c) \leqslant \sigma(a,c)$$

. The morphisms from $\underline{A} = (A, \alpha, \sigma)$ to $\underline{B} = (B, \beta, \tau)$ are the binaries relations R from A to B, verifying :

- 1. $Ra \neq \emptyset$ where $Ra = \{b \in B \mid aRb\}$
- 2. aRb and $a'Rb' \Longrightarrow \sigma(a,a') \leqslant \tau(b,b')$
- 3. aRb and $\alpha(a) \leqslant \tau(b,b') \Longrightarrow aRb'$

. The composition of two morphisms

$$\underline{A} = (A, \alpha, \sigma) \xrightarrow{R} \underline{B} = (B, \beta, \tau) \xrightarrow{S} \underline{C} = (C, \nu, \rho)$$

is the morphism $SR : \underline{A} \longrightarrow \underline{C}$ defined by :

aSRc iff there exists $b \in B$, $c' \in C$ such aRb, bSc' and $\alpha(a) \leqslant \rho(c,c')$.

The identity $1_A : \underline{A} \longrightarrow \underline{A}$ is $a 1_{\underline{A}} a' \Longleftrightarrow \alpha(a) = \sigma(a,a')$.

.R : $\underline{A}$ $\underline{B}$ is a monomorphism iff :

aRb and $a'Rb' \Longrightarrow \sigma(a,a') = \alpha(a) \wedge \alpha(a') \wedge \tau(b,b')$.

G.MYCEK (3) has proved that <u>the category JTF is a topos</u>.

1.<u>Lemma</u> (4) : Given a TFS $\underline{A} = (A, \alpha, \sigma)$, if $a_0, a_1 \in A$, $a_0 \neq a_1$ and $\alpha(a_0) = \sigma(a_0, a_1)$ (i-e, $a_0 \, 1_A \, a_1$) then we can "forget" a_0 . Precisely : let $\underline{A}' = (A', \alpha', \sigma')$ where $A' = A - \{a_0\}$ and $\sigma' = \sigma|_{A' \times A'}$ then $\underline{A}$ is isomorphic to $\underline{A}'$ in JTF.

<u>Definition</u> : Given <u>A</u> = (A, α, σ), a <u>frame</u> of <u>A</u> is an object <u>A'</u> = (A', σ', σ') of JTF such that <u>A'</u> is isomorphic to <u>A</u> and for any a,a' ε A', $a \neq a' \Longrightarrow \sigma'(a,a') < \alpha'(a) \wedge \alpha'(a')$.

2. <u>Proposition</u> : for any <u>A</u> there exist a frame.

The demonstration is in (4). With the axiom of choice, the frame play the role of cardinals number of set.

3. <u>Lemma</u> : given <u>A</u> = (A, α, σ) for a_1, a_2, a_3 ε A, let $r_1 = \sigma(a_2, a_3)$ $r_2 = \sigma(a_1, a_3)$ $r_3 = \sigma(a_1, a_2)$ and suppose that $r_1 \leqslant r_2 \leqslant r_3$, then $r_1 = r_2$.

4. <u>Example</u> : J = (0,0.2,0.4,0.6,0.8,1). A = {a,b,c,d,e,f,g,h,i,j,k}. To recognize the structure of <u>A</u> it is enough to give a table of σ(x,y), x,y ε A, a table of 11 x 11 square. the diagonal gives the α(x) and by reason of symetry, it is sufficent to replenish the inferior triangle. For the first column we can choose the value freely under the condition

$$\sigma(a,x) \leqslant \alpha(a) \wedge \alpha(a),\ x \in A,$$

but when we change the order of elements of A, it is convenient to choose a decreasing sequence. By the lemma 3, the second column is a reproduction of the first one. For the third column, σ(c,d) may be chosen between 0.4 and 0.6, and for the rest we reproduce the second column. The sixth column, σ(f,x) is freely chosen. We can certainly choose it decreasing. The ninth column, σ(i,x) is freely chosen, and all the other columns reproduce the precedent ones.

	a	b	c	d	e	f	g	h	i	j	k
a	1										
b	0.8	1									
c	0.4	0.4	0.8								
d	0.4	0.4	0.6	0.6							
e	0.2	0.2	0.2	0.2	0.4						
f	0	0	0	0	0	0.6					
g	0	0	0	0	0	0.4	0.8				
h	0	0	0	0	0	0.2	0.2	1			
i	0	0	0	0	0	0	0	0	0.6		
j	0	0	0	0	0	0	0	0	0.6	0.6	
k	0	0	0	0	0	0	0	0	0.2	0.2	1

So we see that in <u>A</u> there are three groups of elements of A which are relatively closed to one another : {a,b,c,d,e} , {f,g,h} , {i,j,k} and furthermore, among $\{a,b,c,d\}$ we see that a,b are close one another, and c,d are close.

5. Definition : given $\underline{A} = (A, \alpha, \sigma)$ and $r \in J$ $A_1 \subset A$, we said that A_1 is a indiscernible r-classe iff $A_1 \neq \emptyset$ and for all $a_1, a_1' \in A_1$, $\sigma(a_1, a_1') \geq r$ for all $a_1 \in A_1$, $a \in A - A_1 \Longrightarrow \sigma(a_1, a) < r$.

6. Definition : let $r \in J$, we say that $\underline{A} = (A, \alpha, \sigma)$ is a r-direct sum of $\underline{A}_1 = (A_1, \alpha_1 \sigma)$ and $\underline{A}_2 = (A_2, \alpha_2, \sigma_2)$ and we note $\underline{A} = \underline{A}_1 \underset{r}{\oplus} \underline{A}_2$ if we have:

- $A = A_1 \sqcup A_2$ (i.e $A_1 \subset A$, $A_2 \subset A$, $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A$, $A_1 \neq \emptyset$, $A_2 \neq \emptyset$.)
- $\sigma_i = \sigma|_{A_i \times A_i}$ $i = 1,2$ and ;
- For every $a_1 \in A$ $a_2 \in A_2$ $\sigma(a_1, a_2) = r$.

Remark : we can say that $\underline{A}_1$ and $\underline{A}_2$ are two r-disjoints sub-objets of $\underline{A}$ and if $a_i, a_i' \in A_i$ $(i = 1,2)$, by lemma 3. we have

$$\sigma(a_i, a_i') = \sigma_i(a_i, a_i') \geq r.$$

In an other way, $\underline{A}_i$ are two r-indiscernible classes.

7. Theorem : all J-totally fuzzy sets can be perfectly decomposed in r-direct sum, $r \in J$.

Demonstration : Let $\underline{A} = (A, \alpha, \sigma)$ and $L_r(\underline{A}) = \{a \in A \mid \alpha(a) \geq r\}$, $r \in J$. For each $r \in J$ such as $L_r(\underline{A}) \neq \emptyset$ the function $\sigma : A \times A \longrightarrow J$ inducing an equivalent relation $L_r(\sigma)$ on $L(\underline{A})$:

$$aL_r(\sigma)a' \Longleftrightarrow \sigma(a,a') > r.$$

Remarking that $J = \{0 = r_0 < r_1 < r < .., < r_n = 1\}$. If $\underline{A} \neq \emptyset$, we can certainly suppose that $L_{r_1}(\underline{A}) = A$. So the equivalent relation $L_{r_1}(\sigma)$ determines a partition on $A = L_{r_1}(\underline{A})$:

$A = \bigsqcup_{i \in I} A_i^{r}$, if $a_i \in A_i^r$, $a_j \in A_j^r$ with $i \neq j$, We have $\sigma(a_i, a_j) < r_1$ so $\sigma(a_i, a_j) = 0$. So we obtain a 0-direct sum : $\underline{A} = \underset{\substack{0 \\ i \in I}}{\oplus} \underline{A}_i^r$. More clearly, we apply a pair of parenthesis of level 0 : ${}^{0}(\ ,\ )^{0}$:

$$\underline{A} = {}^{0}\left(\underset{i \in I}{\oplus} \underline{A}_i^{r_1} \right)^{0}$$

where each $\underline{A}_i^{r_1}$ is a r_1-indiscernible classe, so we say that $\underline{A}$ is decomposed in 0-sum of his r_1-indiscernible class. We return take the $\underline{A}_i^{r_1}$ for $\underline{A}$: for each $\underline{A}_i^{r_1}$ such as $L_{r_1}(\underline{A}_i^{r_1}) \neq \emptyset$, the equivalent relation $L_{r_2}(\sigma)$ determin a partition on $L_{r_1}(\underline{A}_i^{r_1})$:

$L_{r_1}(A_i^{r_1}) = \bigsqcup_{j \in J_i} A_{i,j}^{r_2}$. We have for all a_j, $a_j' \in A_{i,j}^{r}$ $\sigma(a_j, a_j') > r_1$ and for all

$a_j \in A^{r_2}_{i,j}$ $a_{j'} \in A^{r_2}_{i,j'}$, $j \neq j'$ $(a_j, a_{j'}) = r_1$.

By the same procedure the r_1-indiscernible class $\underline{A}^{r_1}_i$ is a r_1-direct sum of its r_2-indiscernible classes :

$$\underline{A}^{r_1}_i = \underset{j \in J_i}{\oplus_{r_1}} \underline{A}^{r_2}_{i,j} = {}^1(\oplus_{r_1} \underline{A}^{r_2}_{i,j})^1_{j \in J_i}$$

and so we have :

$$\underline{A} = {}^o(\oplus^1_o(\oplus_{r_1} \underline{A}^{r_2}_{i,j})^1_{j \in J_i})^o_{i \in I}$$

Applying at most n times this procedure, we obtain a complete decomposition of $\underline{A}$ in all its indiscernible classes. For example, we take $\underline{A}$ as in example of 4. We have the following decomposition :

$$\underline{A} = (((\underline{a} \underset{0.8}{\oplus} \underline{b}) \underset{0.4}{\oplus} (\underline{c} \underset{0.6}{\oplus} \underline{d})) \underset{0.2}{\oplus} \underline{e}) \underset{0}{\oplus} ((\underline{f} \underset{0.4}{\oplus} \underline{g}) \underset{0.2}{\oplus} \underline{h}) \underset{0}{\oplus} ((\underline{i} \underset{0.6}{\oplus} \underline{j}) \underset{0.2}{\oplus} \underline{k});$$

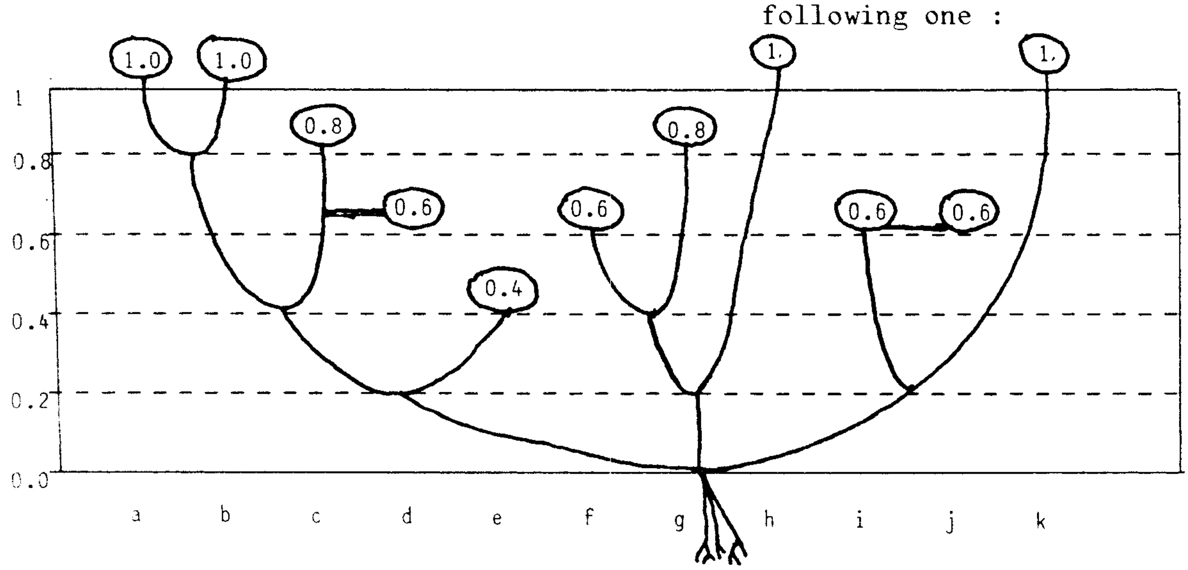
Moreover, it is possible to input this structural information of $\underline{A}$ in a computer in the form :

=((((1)0.8(1))0.4((0.8)0.6(0.6))0.2(0.4))—
—((((0.6)0.4(0.8))0.2(1))...(((0.6)0.6(0.6))0.2(1)).

8. <u>The tree of indiscernible decomposition of a A</u> :

In the demonstration of the theorem 7 we saw that if a, a' ε A $\sigma(a, a') = r_i$ there exist a unique pair of "r_i-level" parenthesis enclosing a and a' : ${}^i(\ldots.a,\ldots,a',\ldots)^i$. so we can draw a tree where the elements of A are considered as leaves with the value $\alpha(x)$. Given two leaves a and a', they are connected by one way by branches and $\sigma(a,a')$ is the value attribued to the lowest branch.

For example, take $\underline{A}$ as in the example of 4 : its tree is the following one :



We remark that :

.we can immediately recognize the structure of $\underline{A}$; $\alpha(x)$ and $\sigma(x,y)$.

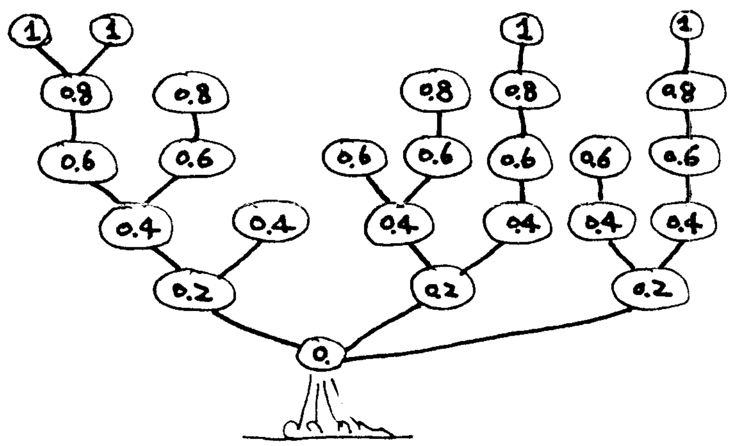
.the elements we can forget are d and j (or d and i) because their are linked by horizontal branches.

.the decomposition scheme clearly appears in the tree.

.we recognize immediately the r-indiscernible classes of A. For all $r \in J$.

.by the lemma 1, we can abandon d and j which are not necessary but we can also fullfill to each level a new leave on the branches. The lemma 1 permits this inverse proceeding : obtaining a full tree of $\underline{A}$;

.According to the meaning of the tree we can explain many elementary notions as morphism, sub-object of a total fuzzy set, equalizer, etc... For example, given $\underline{A}$, let $\mathcal{C}_A = \{$r-indiscernible classe of $\underline{A}$ $r \in J\}$,we have a morphism $R : \underline{A} \longrightarrow \underline{B}$ in bijection with a function $\tilde{R} : \mathcal{C}_A \longrightarrow \mathcal{C}_B$ such that if $x \in \mathcal{C}$ is a r-classe, so $y = \tilde{R}(x) \in \mathcal{C}_B$ is also a r-classe. If $y' = \tilde{R}(x')$ then we have $\underset{\cdot}{\sigma}(x,x') \leqslant \underset{\cdot}{\tau}(y,y')$ ($\underset{\cdot}{\sigma}$ is the prolongation of σ from A to $\mathcal{C}_A$: $A \subset \mathcal{C}_A$ since each $a \in A$ is a $\alpha(a)$-classe

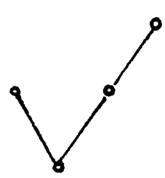


the full tree of $\underline{A}$.

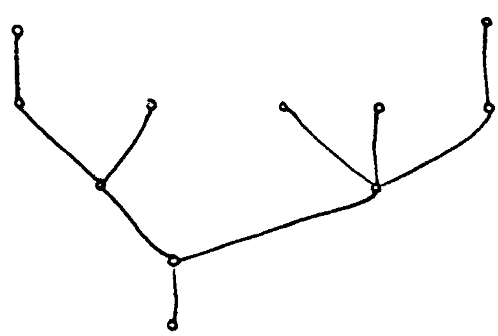
We also can calculate the number n of sub-objects of $\underline{A}$: the tree is a powerful idea to study the structure of any abstract category (4);

. According to the meaning of the tree of $\underline{A}$ we can calculate the number n of all sub-objects of $\underline{A}$:

$n=5$



$n=2\times3=6$



$n=((3\times2)+1)\times((2\times2\times3)+1)+1=92.$

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