4

FUZZY TOPOLOGIES AND TOPOLOGICAL SPACE OBJECTS IN A TOPOS

Ulrich Höhle

Fachbereich Mathematik der Bergischen Universität-Gesamthochschule Wuppertal, Gaußstr. 20, D-5600 Wuppertal 1, Federal Republic of Germany.

Abstract. The purpose of this paper is to point out the link between Lowen's fuzzy topologies and topological space objects in the topos L-SET . As the main result we obtain the following theorem : Lowen's fuzzy topologies are the external version of the internal topologies in L-SET .

Keywords. Complete Heyting algebras, equality relations, L-valued sets, fuzzy topologies, topological space objects.

For the non-topos theorist we include a section on fundamental properties of the category L-SET, which are significant for a clear mathematical understanding of fuzzy set theory.

1. L-fuzzy subsets and subobjects in the category L-SET

Let $(L,\leq)$ be a complete Heyting algebra and X be an ordinary set; a L-equality relation on X is a map $E : X \times X \longrightarrow L$ satisfying the following conditions

(E1)	$E(x,y) \leq E(y,x)$	$\forall\ y,x \in X$	(Symmetry)
(E2)	$E(x,y) \wedge E(y,z) \leq E(x,z)$	$\forall\ x,y,z \in X$	(Transitivity)

The value $E(x,y)$ can be interpreted as the degree that the elements x and y are equal . In this context we accept the requirement that two elements can only be equal if they exist ; thus $E(x,x)$ is the degree of existence of x .

If E is a L-equality relation on X, then the ordered pair (X,E) is called a L-valued set. A typical example of a L-valued set is the following one : $X = L$, $E(\alpha,\beta) = \alpha \Leftrightarrow \beta$, $\alpha \Leftrightarrow \beta = (\bigvee\{\lambda \in L , \alpha \wedge \lambda \leq \beta\}) \wedge (\bigvee\{\lambda \in L , \beta \wedge \lambda \leq \alpha\})$. $(L, \Leftrightarrow)$ is a L-valued set and the degree of existence of every element $\alpha \in L$ is equal to $\mathbb{1}$ (= greatest element in L) .

The category L-SET consists of these data (cf. [2], [3]) : Objects are L-valued sets and morphisms $M : (X,E) \longrightarrow (Y,F)$ are ordinary maps $M : X \times Y \longrightarrow L$ satisfying the conditions

(M1) $M(x,y) \wedge E(x,\hat{x}) \leq M(\hat{x},y)$
$M(x,y) \wedge F(y,\hat{y}) \leq M(x,\hat{y})$ (Extensionality)

(M2) $M(x,y) \leq E(x,x) \wedge F(y,y)$ (Strictness)

(M3) $M(x,y) \wedge M(x,\hat{y}) \leq F(y,\hat{y})$ (Single-valued)

(M4) $\bigvee\{M(x,y) , y \in Y\} = E(x,x)$ (Everywhere def.)

The composition of $M_1 : (X,E) \longrightarrow (Y,F)$ and $M_2 : (Y,F) \longrightarrow (Z,G)$ is defined by $M_2 \circ M_1(x,z) = \bigvee\{M_1(x,y) \wedge M_2(y,z) , y \in Y\}$. The identity of (X,E) is E .

It is well known that L-SET is a topos (cf. [2]) ; the subobject classifier is the object $(L, \Leftrightarrow)$ together with the arrow 'true' $t : \mathbf{1} \hookrightarrow (L, \Leftrightarrow)$ determined by $t(\cdot,\lambda) = \lambda$. Moreover every morphism $M : (X,E) \longrightarrow (L, \Leftrightarrow)$ can externally identified with an ordinary map $f : X \longrightarrow L$ satisfying the following conditions

(S1) $f(x) \wedge E(x,y) \leq f(y)$ (Extensionality)
(S2) $f(x) \leq E(x,x)$ (Strictness) ;

in particular M is given by $M(x,\lambda) = E(x,x) \wedge (f(x) \Leftrightarrow \lambda)$. Thus the power object $P(X,E)$ consists of the set $\mathbb{E}(X)$ of all extensional and strict maps $f : X \longrightarrow L$ and the equality relation $[\![\, ,]\!]$ defined by $[\![g,f]\!] = \bigwedge_{x \in X} (f(x) \Leftrightarrow g(x))$.

If E_c is the crisp equality relation on X - i.e. $E_c(x,y) = \begin{cases} \mathbb{1} , & x = y \\ 0 , & x \neq y \end{cases}$, then every morphism

$M : (X,E_c) \longrightarrow (L, \Leftrightarrow)$ can be identified with a <u>L-fuzzy subset</u> of X in the sense of Goguen [1] ; hence the external version of the power object $P(X,E_c)$ coincides with L^X . In this context the reader may notice the fundamental fact : If $L \neq 2$, then the power object $P(X,E_c)$ of a crisp L-valued set is <u>never crisp</u> - i.e. $[\![\ ,\ ]\!]$ is always different from the crisp equality E_c ; this has of course consequences in the definition of a topology in the framework of L-SET .

Since L-SET is a topos, for every monomorphism $M : (X,E) \hookrightarrow (Y,F)$ there exists a unique morphism (so-called characteristic morphism) $\chi_M : (Y,F) \longrightarrow (L, \Leftrightarrow)$ s.t. the diagram

$$\begin{array}{ccc} (X,E) & \xrightarrow{\ !\ } & \mathbb{1} \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle t} \\ (Y,F) & \xrightarrow[\chi_M]{} & (L, \Leftrightarrow) \end{array}$$

is a pullback, and vice-versa every morphism $\chi : (Y,E) \longrightarrow (L, \Leftrightarrow)$ induces in this way a unique subobject of (Y,F). In this sense subobjects of (Y,F) and morphisms $\chi : (Y,F) \longrightarrow (L, \Leftrightarrow)$ are equivalent concepts. The <u>subobject</u> of (Y,E_c) corresponding to a <u>L-fuzzy subset</u> $f : Y \longrightarrow L$ is given by

$X = \{ s \in L^Y \mid s(y) \wedge s(x) = 0 \text{ for } y \neq x \ , \ s(y) \leq f(y) \ \forall\, y \in Y \}$

$E(s_1,s_2) := \bigvee \{ s_1(y) \wedge s_2(y) \ , \ y \in Y \}$

$M : (X,E) \hookrightarrow (Y,E_c) \quad , \quad M(s,y) = s(y) \quad \forall\, s \in X \quad \forall\, y \in Y$.

As in every category with a terminal object $\mathbb{1}$ a point of (X,E) is an arrow $D : \mathbb{1} \hookrightarrow (X,E)$ - i.e. $D : X \longrightarrow L$ satisfies the following conditions (cf. (M1) - (M4))

$D(x) \wedge E(x,y) \leq D(y)$

$D(x) \wedge D(y) \leq E(x,y)$

$\bigvee \{ D(x) \ , \ x \in X \} = \mathbb{1}$

A point D of (X,E) is a member of a subobject $M : (Y,F) \hookrightarrow (X,E)$ (notation : $D \in M$) iff there exists $K : \mathbb{1} \hookrightarrow (Y,F)$ s.t. $D = M \circ K$. If f_M is the extensional and strict L-fuzzy subset of X corresponding to the subobject M , then the following assertions are equivalent :

(i) $D \in M$

(ii) $D(x) \leq f_M(x) \quad \forall\, x \in X$.

Finally we remark that for every point D of the power object $P(X,E) = (\mathbb{E}(X), [\![\ ,\ ]\!])$ there exists a unique element $f_0 \in \mathbb{E}(X)$ inducing D - i.e. $D(f) = [\![f_0, f]\!] \quad \forall\, f \in \mathbb{E}(X)$.

2. Topological space objects in L-SET

First we recall the definition of a topological space object in a topos given by L.N. Stout (cf. [6]) : Let $\mathcal{E}$ be a topos A an object of $\mathcal{E}$, $P(A)$ be the power object of A and let (Ω, t) be the subobject classifier. We denote by $ev_A : P(A) \times A \longrightarrow \Omega$ (resp. $ev_{P(A)} : P^2(A) \times P(A) \longrightarrow \Omega$) the evaluation arrow and by π_{12} (resp. π_{23}, π_{13}) the projection from $P^2(A) \times P(A) \times A$ onto $P^2(A) \times P(A)$ (resp. $P(A) \times A$, $P^2(A) \times A$) . Further let $M : Y \hookrightarrow P^2(A) \times P(A) \times A$ the subobject corresponding to the characteristic morphism $ev_{P(A)} \circ \pi_{12} \wedge ev_A \circ \pi_{23}$: Then the <u>union map</u> $\underline{U} : P^2(A) \longrightarrow P(A)$ is the exponential adjoint of the characteristic morphism of the subobject $\exists_{\pi_{13}}(M)$.

A <u>topological space object</u> in $\mathcal{E}$ is a pair (A, T_A) where T_A is a subobject of $P(A)$ satisfying the following axioms

($\mathcal{O}$1) $\quad \emptyset \in T_A \quad , \quad A \in T_A$

($\mathcal{O}$2) $\quad B \in T_A$ and $B' \in T_A$ implies $B \wedge B' \in T_A$

($\mathcal{O}$3) $\quad$ If S is a subobject of $P(A)$ with $S \leq T_A$, then $\underline{U} \circ \ulcorner \chi_S \urcorner \in T_A$, where $\ulcorner \chi_S \urcorner$ is the name of the characteristic morphism of S .

Since in L-SET the union map is induced by an ordinary map $\Sigma : \mathbb{E}^2(X) \longrightarrow \mathbb{E}(X)$ determined by

$$\Sigma(\mu)(x) = \bigvee_{f \in \mathbb{E}(X)} (\mu(f) \wedge f(x)) \qquad \mu \in \mathbb{E}^2(X) ,$$

the axioms of a topological space object in L-SET can be rewritten as follows : Let (X,E) be a L-valued set, $P(X,E) = (\mathbb{E}(X), [\![\ ,\ ]\!])$ the power object of (X,E) and let μ be a L-fuzzy subset of $\mathbb{E}(X)$. The triple (X,E,μ) is a topological space object in L-SET iff μ satsifies the following conditions

$(\mathcal{O}0)$ $\mu(f) \wedge [\![f,g]\!] \leq \mu(g)$ (Extensionality)

$(\mathcal{O}1)$ $\mu(1_\emptyset) = \mu(1_E) = \mathbb{1}$ where $1_E(x) = E(x,x)$

$(\mathcal{O}2)$ $\mu(f) = \mathbb{1}$ and $\mu(g) = \mathbb{1} \Longrightarrow \mu(f \wedge g) = \mathbb{1}$

$(\mathcal{O}3)$ $\forall\ \nu \in \mathbb{E}^2(X)$ with $\nu \leq \mu$: $\mu(\Sigma(\nu)) = \mathbb{1}$

A <u>L-fuzzy topological space</u> $(X,E,\mathcal{O})$ consits of a L-valued set (X,E) and an ordinary subset $\mathcal{O}$ of $\mathbb{E}(X)$ equipped with the subsequent properties

$(\hat{\mathcal{O}}1)$ $\alpha_E \in \mathcal{O}\ \ \forall\ \alpha \in L$, where $\alpha_E(x) = E(x,x)$ (Constants Cond.)

$(\hat{\mathcal{O}}2)$ $f, g \in \mathcal{O} \Longrightarrow f \wedge g \in \mathcal{O}$

$(\hat{\mathcal{O}}3)$ $\mathcal{T} \subseteq \mathcal{O} \Longrightarrow \bigvee\{g,\ g \in \mathcal{T}\} \in \mathcal{O}$

If we replace the Heyting algebra $(L, \leq)$ by the real unit interval and the equality relation by the crisp equality E_c , then $(\hat{\mathcal{O}}1)$ - $(\hat{\mathcal{O}}3)$ are just the axioms of a fuzzy topological space in the sense of R. Lowen (cf. [4],[5]). In the following we show that L-fuzzy topological spaces are the external version of topological space objects in L-SET .

<u>2.1 Proposition</u> If (X,E,μ) is a topological space object , then $\mathcal{O}_\mu := \{ g \in \mathbb{E}(X) ,\ \mu(g) = \mathbb{1} \}$ is a L-fuzzy topology on (X,E) , and μ is given by

$\mu(f) = [\![\mathring{f}, f]\!]$ where $\mathring{f} = \bigvee \{ g \in \mathcal{O}_\mu ,\ g \leq f \}$.

<u>Proof</u>. The axioms $(\hat{\mathcal{O}}2)$ and $(\hat{\mathcal{O}}3)$ are evident. In order to verify $(\hat{\mathcal{O}}1)$ we first observe :

$\alpha \wedge [\![f,\alpha_E]\!] \leq [\![1_E, \alpha_E]\!] \wedge [\![\alpha_E, f]\!] \leq \mu(f) \quad \forall\ f \in \mathbb{E}(X)$.

Now we put $\nu(f) = \alpha \wedge [\![f,\alpha_E]\!]$ and apply $(\mathcal{O}3)$: $\mu(\Sigma(\nu)) = \mathbb{1}$ - i.e. $\alpha_E \in \mathcal{O}_\mu$.

Further we obtain from $\nu(f) := \mu(h) \wedge [\![h,f]\!] \leq \mu(f)$

that $\mu(\Sigma(\nu)) = \mathbb{1}$ - i.e. $\mu(h) \wedge h(\cdot) \in \mathcal{O}_\mu \quad \forall\, h \in \mathbb{E}(X)$.
Therewith the inequality $\mu(h) \leq [\![\mathring{\hat{h}}, h]\!]$ is valid - i.e.
$\mu(h) = [\![\mathring{\hat{h}}, h]\!]$.

2.2 Proposition. Let $(X,E,\mathcal{O})$ be a L-fuzzy topological space.
Then the L-fuzzy subset $\mu_{\mathcal{O}}$ of $\mathbb{E}(X)$ defined by
$\mu_{\mathcal{O}}(f) = [\![\mathring{\hat{f}}, f]\!]$ where $\mathring{\hat{f}} = \vee\{g \in \mathcal{O},\ g \leq f\}$
satisfies $(\mathcal{O}0)$ - $(\mathcal{O}3)$ -i.e. $(X,E,\mu_{\mathcal{O}})$ is a topological
space object in L-SET .

Proof. First we show that $\mu_{\mathcal{O}}$ is extensional - i.e. $\mu_{\mathcal{O}}$ fulfills the axiom $(\mathcal{O}0)$:

$[\![\mathring{\hat{h}}, h]\!] \wedge [\![h, f]\!] \wedge f(x) \leq [\![h, f]\!] \wedge \mathring{\hat{h}}(x) \leq f(x) \quad \forall\ x \in X$

- i.e. $\mu(h) \wedge [\![h, f]\!] \leq \mu(f)$.

The axioms $(\mathcal{O}1)$ and $(\mathcal{O}2)$ are evident . Therefore we only verify $(\mathcal{O}3)$: If $\nu(f) \leq [\![\mathring{\hat{f}}, f]\!] \ \forall f \in \mathbb{E}(X)$, then $\nu(f) \wedge f(x) \leq \nu(f) \wedge \mathring{\hat{f}}(x)$; thus $\Sigma(\nu) \in \mathcal{O}$ - i.e. $\mu_{\mathcal{O}}(\Sigma(\nu)) = \mathbb{1}$.

2.3 Corollary. Let (X,E,μ) be a topological space object in L-SET . Then the subobject $M : (Y,F) \hookrightarrow P(X,E)$ corresponding to μ is given by

$Y = \mathcal{O}_\mu$, $F(f,g) = [\![f, g]\!] \quad \forall\ f, g \in \mathcal{O}_\mu$

$M : (Y,F) \hookrightarrow P(X,E)$, $M(f,h) = [\![f, h]\!]$.

Summing up L-fuzzy topologies are just the e x t e r n a l description of the i n t e r n a l topologies in L-SET . In this context it is important to note that the "Constants Condition" in Lowen's axiom system is a consequence of the categorial formulation of the topological axioms ; more precisely the "Constants Condition" is just that condition which permits to internalize L-fuzzy topologies in the category L-SET .
Moreover, if (X,E,μ) is a topological space object , then μ can be considered as a L-fuzzy openness operator on $\mathbb{E}(X)$ - i.e. $\mu(f)$ is the degree that the L-fuzzy subset f is open. Thus the axioms $(\mathcal{O}1)$ - $(\mathcal{O}3)$ are precisely the f u z z i f i c a t i o n of the usual topological axioms .

Acknowledgement. I am very much obliged to Umberto Cerruti having drawn my attention to topos theory .

References.

[1] Goguen, J.A., L-fuzzy sets, J. Math. Anal. Appl. 18 (1967), 145 - 174 .

[2] Goldblatt,R., Topoi, the categorial analysis of logic (North-Holland, Amsterdam, 1979) .

[3] Higgs, D., A categorial approach to Boolean-valued set theory , 1973 (unpublished) .

[4] Lowen, R., Topologies floues, C. R. Acad. Sci. (Paris), 278 (1974) , 925 -928 .

[5] Lowen, R., Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976), 621 -633 .

[6] Stout, L.N Topological space objects in a topos II : $\mathcal{E}$-Completeness and $\mathcal{E}$-Cocompleteness manuscripta math. 17 (1975) , 1 - 14 .