ON FUZZY PROGRAMMING

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Introduction. A theory of fuzzy programming presented in this paper is quite different from the well-known fuzzy linear programming theory developed by Hannan [2], Rödder [4] and Zimmermann [5] . Using the algebraic apparatus developed in [3] it is possible to give a complete theory of fuzzy programming (e.g. [1]). At first the fuzzy programming problems are defined and described and then the fundamental duality theorem is formulated and proved. Assuming the duality theorem we prove the important "equilibrium theorem" .

Using the algebraic machinery developed in the paper [3] we are going to formulate the fuzzy programming problem FP. In the fuzzy vector matrix notation which we have introduced, this problem can be written in the following form:

Gieven an $m \times n$ fuzzy matrix $A = (\{x_{ij}, a_{ij}\})$, an n- fuzzy vector $b = (\{y_j, b_j\})$, and an m-fuzzy vector $c = (\{z_i, c_i\})$, find an m-fuzzy point $\{a^*\}_P$ with the nonnegative support such that

$$\{a^*\}_P \cdot c \geqslant \{a\}_P \cdot c \quad , \quad \forall a \in R^m_+ \qquad (1)$$

where P is the fuzzy subset of solution of fuzzy inequality

$$\{a\} \cdot A \leqslant b \qquad (2)$$

and $\{a\}_P = (\{a_i, \mu_P(a)\})$, $a = (a_1, \ldots, a_m) \in R^m$.

Note that $\{a\}_P$ is the singleton fuzzy subset.

A fuzzy programming problem FP is called feasible if there exists a fuzzy point $\{a\}_P$ such that $a \in R^m_+$ and $\{a\}_P \neq \emptyset$. Such a fuzzy point is called feasible solution. A feasible fuzzy point $\{a^*\}_P$ satisfying the condition (1) is called an optimal solution.

The dual problem FP' is then that of finding n-fuzzy point $\{d^*\}_{P'}$ with the nonnegative support such that

$$\{d^*\}_{P'} \cdot b \leqslant \{d\}_{P'} \cdot b \quad , \qquad \forall d \in R^n_+ \qquad (1')$$

where P' is the fuzzy subset of solution of fuzzy inequality

$$A \cdot \{d\} \geqslant c \qquad (2')$$

Presented above fuzzy programming problem FP is called a standard FP. We can distinquish the various types of FP according to the nature of the contraints and of the optimal solution.

DEFINITION 1. Let A, b and c be as before. The canonical fuzzy programming problem FP is that of finding a fuzzy point

$\{a^*\}_P$ with the nonnegative support such that

$$\{a^*\}_P \cdot c \geqslant \{a\}_P \cdot c \qquad \forall\, a \in R_+^m \ , \tag{3}$$

where P is the fuzzy subset of solution of fuzzy equation

$$\{a\} \cdot A = b \tag{4}$$

Analogously, we define the canonical fuzzy problem FP , which is that of finding a fuzzy point $\{a^*\}_P$ with the nonnegative support that $\forall a \in R_+^m$

$$\{a^*\}_P \cdot c \leqslant \{a\}_P \cdot c \quad .$$

We shall now show that the standard and canonical fuzzy problems are equivalent in the sence that either one can be transformed into the other in an obvious manner.

If we replace (4) above by the inequalities

$$\begin{gathered} \{a\} \cdot A \leqslant b \\ (-1) * \{a\} \cdot A \cdot (-1) \dot{\leqslant} (-1) * b \cdot (-1) \end{gathered} \tag{5}$$

it is clear that we have rephrased the canonical problem as a standard problem. On the other hand, to change a standard fuzzy problem to a canonical one we replace the fuzzy inequality

$$\{a\} \cdot A \leqslant b \tag{6}$$

by the equation

$$\{\bar{a}\} \cdot \bar{A} = b \ , \tag{7}$$

where $\{\bar{a}\}$ and $\bar{A}$ are the fuzzy point and the fuzzy matrix which we defined in [3] .

The condition (1) we replace by the condition

$$\{\bar{a}^*\}_{\bar{P}} \cdot \bar{c} \geqslant \{\bar{a}\}_{\bar{P}} \cdot \bar{c} \ , \tag{8}$$

where $\bar{c} = (\{\bar{z}_i, \bar{c}_i\})$ is an $(m+n)$ fuzzy vector such that $\forall\, i = 1(1)m \quad \bar{z}_i = z_i \ , \ \bar{c}_i = c_i$ and $\forall\, i = (m+1)(1)m+n \quad \bar{z}_i = 0, \ \bar{c}_i = 1.$

We now have a fuzzy problem FP whose contraints are fuzzy equations, and clearly $\{\bar{a}\}$ has a nonnegative support and $\{a'\}_P$

subject to (1) and (6) iff $\{\bar{a}'\}_{\bar{P}}$ subject to (7) and (8), where $\bar{P}$ is fuzzy subset of solution of fuzzy equation (7).

The most general case FP is problem in which some $\{a\}$ have arbitrary support and where the contraints include both equations and inequalities.

DEFINITION 2. Let $a^1 = (\{x_{i1}, a_{i1}\}), \dots, a^r = (\{x_{ir}, a_{ir}\})$, $d^1 = (\{w_{i1}, d_{i1}\}), \dots, d^s = (\{w_{is}, d_{is}\})$ and $c = (\{z_i, c_i\})$ are m-fuzzy vectors and let $\{h_1, k_1\}, \dots, \{h_r, k_r\}, \{l_1, f_1\}, \dots, \{l_s, f_s\}$ are the fuzzy elements. Let $\forall j = 1(1)s$

$$f_j / \mathop{\top}_{i=1}^{m} d_{ij} = \beta \in \langle 0,1 \rangle .$$

The general FP is that of finding an m-fuzzy point $\{a^*\}_P$ such that

$$\{a^*\}_P \cdot c \succcurlyeq \{a\}_P \cdot c \quad , \qquad \forall\, a \in R^m \tag{9}$$

where P is the fuzzy subset of solution of the

$$\begin{aligned} \{a\} \cdot a^j &\leqslant \{h_j, k_j\} \qquad \text{for } j = 1(1)r , \\ \{a\} \cdot d^p &= \{l_p, f_p\} \qquad \text{for } p = 1(1)s . \end{aligned} \tag{10}$$

It is clear that both the standard and canonical FPs are special cases of the general FP. Conversely, the general FP can transform into either of the others. To obtain a standard problem equivalent to the general problem above, we first obtain contraints involving only fuzzy inequalities by replacing the fuzzy equations in (10) above by the inequalities

$$\begin{aligned} &\{a\} \cdot d^p \leqslant \{l_p, f_p\} , \\ &(-1) * \{a\} \cdot d^p \cdot (-1) \; \dot{\leqslant} \; (-1) * \{l_p, f_p\} \cdot (-1) . \end{aligned} \tag{11}$$

Next, introduce new unknown n-fuzzy points $\{a'\}$ and $\{a''\}$ with the nonnegative supports and replace fuzzy inequalities (10) and (11) by

$$(\{a'\} - \{a''\})\cdot a^j \leq \{h_j, k_j\},$$
$$(\{a'\} - \{a''\})\cdot d^p \leq \{l_p, f_p\}, \qquad (12)$$
$$(-1)*(\{a'\} - \{a''\})\cdot d^p\cdot(-1) \lesssim (-1)*\{l_p, f_p\}\cdot(-1)$$

and require that $\forall a', a'' \in R^m_+$

$$(\{a'^*\}_P - \{a''^*\}_P)\cdot c \geq (\{a'\}_P - \{a''\}_P)\cdot c \qquad (13)$$

The fuzzy points $\{a'^*\}_P$ and $\{a''^*\}_P$ satisfy the condition (13) subject (12) iff the fuzzy point $\{a^*\}_P = \{a'^*\}_P - \{a''^*\}_P$ solves the original problem.

Lemma 1. Let $\{a\}_P$ and $\{d\}_{P'}$ are the feasible solutions of the standard fuzzy programmings FP and FP' respectively. Then

$$\{a\}_P\cdot c \leq \{d\}_{P'}\cdot b\,. \qquad (14)$$

Proof. From (2) wehave: $\forall j$

$$\sum_i \{x_{ij}, a_{ij}\}\cdot\{a_i, \mu_P(a)\} \leq \{y_j, b_j\} \qquad (*)$$

and from (2) we have: $\forall i$

$$\sum_j \{x_{ij}, a_{ij}\}\cdot\{d_j, \mu_{P'}(d)\} \gtrsim \{z_i, c_i\} \qquad (**)$$

Multiplying (*) by $\{d_j, \mu_{P'}(d)\}$ and summing on j, and multiplying (* *) by $\{a_i, \mu_P(a)\}$ and summing on i gives

$$\sum_i \{a_i, \mu_P(a)\}\cdot\{z_i, c_i\} \leq \sum_j \{d_j, \mu_{P'}(d)\}\cdot\{y_j, b_j\}$$

which means

$$\{a\}_P\cdot c \leq \{d\}_{P'}\cdot b\,.$$

Theorem 1. (optimality criterion). If there exist feasible solutions $\{a'\}_P$ and $\{d'\}_{P'}$ of the standard fuzzy programmings FP and FP' respectively such that

$$\{a'\}_P\cdot c = \{d'\}_{P'}\cdot b \qquad (15)$$

then these feasible solutions are, in fact, optimal solutions of their respective problems.

Proof. Let $\{a\}_P$ be any other feasible solution of the FP. Them from the Lemma 1

$$\{a\}_P \cdot c \leq \{d'\}_{P'} \cdot b ,$$

and combining this with (15) gives

$$\{a\}_P \cdot c \leq \{a'\}_P \cdot c$$

showing that $\{a'\}_P$ is an optimal solution. A symmetrical argument proves the optimality of $\{d'\}_{P'}$.

Let us note that theory of fuzzy programming presented in this paper is true for first and second kind inequalities. Unfortunately the fundamental duality theorem is true iff in the fuzzy programmings FP and FP' we have the second kind inequalities.

Theorem 2. (fundamental duality theorem). If both a fuzzy program and its dual are feasible and are described by the second kind inequalities then both have optimal solutions and the values of the two fuzzy programs are the same. If either fuzzy program is infeasible them neither has an optimal solution.

Proof. From classical fundamental duality theorem we have that supp $(\{a^*\}_P \cdot c) = \text{supp}\,(\{d^*\}_{P'} \cdot b)$.

We are going to prove that

$$\mathop{\top}_{i=1}^{m} c_i \cdot \mu_P(a^*) = \mathop{\top}_{j=1}^{n} b_j \cdot \mu_{P'}(d^*) .$$

Because $\{a^*\}_P$ and $\{d^*\}_{P'}$ are the optimal solutions of FP and FP' respectively, then there exist i and j such that

$$\mu_P(a^*) = b_j \,/ \mathop{\top}_{i',j'} a_{i'j'}$$

and

$$\mu_{P'}(d^*) = c_i \,/ \mathop{\top}_{i',j'} a_{i'j'} .$$

Note that, then

$$b_j = \min_{j'} b_{j'} \quad \text{and} \quad c_i = \min_{i'} c_{i'} .$$

So,

$$\mathop{\top}_{i'=1}^{m} c_{i'} \cdot \mu_P(a^*) = c_i \cdot b_j / \mathop{\top}_{i',j'} a_{i'j'}$$

and

$$\mathop{\top}_{j'=1}^{n} b_{j'} \cdot \mu_{P'}(d^*) = b_j \cdot c_i / \mathop{\top}_{i',j'} a_{i'j'} .$$

From classical fundamental duality theorem it follows that the second part of this theorem is true.

Theorem 3. (canonical equilibrium theorem). Let $\{a^*\}_P$ be a solution with the nonnegative support of fuzzy equation

$$\{a\} \cdot A = b \qquad (16)$$

and let $\{d\}_{P'}$ be a solution of second kind inequalities

$$A \cdot \{d\} \geqslant c .$$

Then

$$\{a^*\}_P \cdot c \geqslant \{a\}_P \cdot c \quad , \qquad \forall a \in R^m_+$$

and

$$\{d^*\}_{P'} \cdot b \leqslant \{d\}_{P'} \cdot b \qquad \forall d \in R^n \qquad (17)$$

iff

$$a^*_i = 0 \quad \text{whenever} \quad A_i \cdot \{d\} > \{z_i, c_i\} ,$$

where A_i is the i-th row of A.

Proof. If (17) satisfied then we have

$$\{a^*\}_P \cdot c = \sum_i \{a^*_i, \mu_P(a^*)\} \cdot \{z_i, c_i\} = \sum_i \{a^*_i, \mu_P(a^*) \cdot A_i \cdot \{d^*\}_{P'} =$$

$$(\sum_i \{a^*_i, \mu_P(a^*) \cdot A_i) \cdot \{d^*\}_{P'} = \{a^*\}_P \cdot A \cdot \{d^*\}_{P'} = b \cdot \{d^*\}_{P'} \qquad (18)$$

and therefore $\{a^*\}_P$ and $\{d^*\}_{P'}$ are optimal solutions. Conversely, if $\{a^*\}_P$ and $\{d^*\}_{P'}$ are optimal then (18) follows from the duality theorem, from which we get

$$\sum_i \{a^*_i, \mu_P(a^*)\} \cdot (A_i \cdot \{d\}_{P'} - \{z_i, c_i\}) = e$$

and since all terms in this sum have nonnegative support, condition (17) follows.

Theorem 4. (equilibrium theorem for standard FP). The feasible solutions $\{a\}_P$ and $\{d\}_{P'}$ of second kind inequality

$$\{a\}\cdot A \leqslant b$$

and of second kind inequality

$$A\cdot\{d\} \geqslant c ,$$

respectively are optimal solutions iff

$$d_j = 0 \quad \text{whenever } \{a\}\cdot A^j < \{y_j, b_j\} \tag{19}$$

and

$$a_i = 0 \quad \text{whenever } A_i\cdot\{d\} > \{z_i, c_i\} , \tag{20}$$

where A^j is the j-th column of A and A_i is the i-th row of A.

Proof. First suppose conditions (19) and (20) hold. Then we have

$$\{d\}\cdot b = \sum_j \{d_j, \mu_{P'}(d)\}\cdot\{y_j, b_j\} = \sum_j \{d_j, \mu_{P'}(d)\}\cdot\{a\}\cdot A^j =$$

$$= \sum_j \{d_j, \mu_{P'}(d)\}\cdot \sum_i \{a_i, \mu_P(a)\}\cdot\{x_{ij}, a_{ij}\} .$$

$$\{a\}\cdot c = \sum_i \{a_i, \mu_P(a)\}\cdot\{z_i, c_i\} = \sum_i \{a_i, \mu_P(a)\}\cdot A_i\cdot\{d\} =$$

$$= \sum_i \{a_i, \mu_P(a)\}\cdot \sum_j \{d_j, \mu_{P'}(d)\}\cdot\{x_{ij}, a_{ij}\} .$$

From this qualities it follows, that $\{d\}\cdot b = \{a\}\cdot c$.
Whence from Theorem 1 $\{a\}_P$ and $\{d\}_{P'}$ are optimal solutions.

Conversely, let $\{a\}_P$ and $\{d\}_{P'}$ are optimal solutions. Then from the duality theorem we have

$$\{a\}_P\cdot c = \{d\}_{P'}\cdot b .$$

So,

$$\sum_i \{a_i, \mu_P(a)\}\cdot\{z_i, c_i\} = \sum_i \{a_i, \mu_P(a)\}\cdot\sum_j \{d_j, \mu_{P'}(d)\}\cdot$$

$$\cdot\{x_{ij}, a_{ij}\} = \sum_j \{d_j, \mu_{P'}(d)\}\cdot\{y_j, b_j\} .$$

From the first equation we have

$$\sum_i \{a_i, \mu_P(a)\} (\{z_i, c_i\} - \sum_j \{d_j, \mu'_P(d)\} \cdot \{x_{ij}, a_{ij}\}) = e$$

but since $\{d\}_{P'}$ is feasible it follows that

$$\text{supp} (\{z_i, c_i\} - \sum_j \{d_j, \mu'_P(d)\} \cdot \{x_{ij}, a_{ij}\})$$

is nonpositive and hence for each i

$$\{z_i, c_i\} - \sum_j \{d_j, \mu'_P(d)\} \cdot \{x_{ij}, a_{ij}\}) = e_i$$

from which (20) follows at once. A symmetrical argument proves condition (19).

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