ON HANDLING FUZZINESS AND RANDOMNESS IN FUZZY RELATIONAL MODELS

Witold Pedrycz
Department of Automatic Control and Comp.Sci.
Silesian Technical University
44-100 Gliwice,Poland

Abstract We deal with a class of fuzzy models using fuzzy relation equations that enable us to handle fuzziness and randomness in a unique manner.A·factor of randomness is put as an underlying structure of the fuzzy equation of the model.

1. Preliminaries

It is evident that fuzziness,as handled by fuzzy set theory,is related with aspects of uncertainty that appear in categories of ill--defined ambiguous concepts.Contrary to it probabilistic form of uncertainty is directly dedicated to description of occurrence or non-occurrence of well defined events.In human activity,for instance in decision-making,both factors are present.Consider for example the following statement,

Within <u>few</u> weeks we can expect a <u>warm</u> season.

It contains judgement that is formulated with the aid of subjective and objective forms of uncertainty.The phenomen being a subject of this statement,is in a certain sense individual one.Moreover there are also some past data,mainly numerical,that may be performed in a framework of probability theory.This is also true in medical diagnosis.Usually there exists a remarkable amount of data/past records of patients/ which reflect relationships between symptoms and diseases/correlation coefficients,regression lines etc./.Nevertheless every situation/a patient examined by a clinician/should be treated as a unique one with some links with probabilistic layer.

The aim of the paper is to propose a form of the fuzzy model which is capable to handle both sources of information/viz.fuzzy and probabilistic/.Due to advanced theory and methods of fuzzy relation equations and their role in system modelling,e.g. [2-5] we follow this way introducing several significant modifications.We apply state approach to system analysis.For concise presentation we will play with discrete universes of control/input/ and state, $\mathbb{U}=\{u_1,u_2,\dots,u_m\}$, $\mathbb{X}=\{x_1,x_2,\dots,x_n\}$

assuming that relationships between input and state viewed as fuzzy sets are described by discrete-time fuzzy relation equation of the first order,

$$X_{k+1}=U_k \circ X_k \circ R, \qquad /1/$$

$U_k \in F(\mathbb{U}), X_k, X_{k+1} \in F(\mathbb{X}), R \in F(\mathbb{U} \times \mathbb{X} \times \mathbb{X})$,k=0,1,2,... .

Model description

As mentioned above the relationships between state and control in discrete time moments are handled by means of fuzzy relation equation/1/.Moreover we have at our disposal some probabilistic characteristics of the system under consideration.They are given in the form of joint probabilities of the triples $(u_l, x_j, y_i), u_l \in \mathbb{U}$, $x_j, y_i \in \mathbb{X}$,

$$p(y_i, x_j, u_l) \quad , \qquad /2/$$

such that

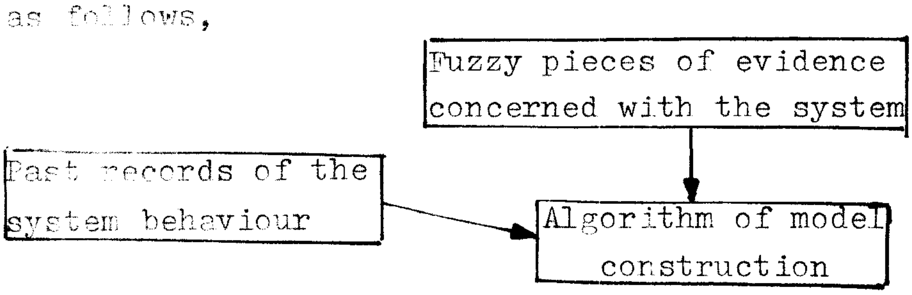
$$\sum_{i,j,l} p\ y_i, x_j, u_l \qquad /3/$$

is satisfied.Conditional probabilities $p(y_i | x_j, u_l)$ express a strength of ties between the respective elements of $\mathbb{X}$ and $\mathbb{X} \times \mathbb{U}$/strength of transition between respective pairs (x_j, u_l) and y_i / and they are calculated according to obvious formula,

$$p(y_i | x_j, u_l) = \frac{p(y_i, x_j, u_l)}{p(x_j, u_l)} \qquad /4/$$

with $p(x_j, u_l)$ being margin probability function ,$p(x_j, u_l) = \sum_i p(y_i, x_j, u_l)$

Note that the fuzzy relation equation embrasses a knowledge about behaviour of the system in"global"-we know how fuzzy labels are related together.Probabilistic knowledge gives us an impression about"local" character of relationships between discrete values of the universes of discourse.It allows us to have any insight into the structure of dependencies between elements in statistical sense.The two sources of information that take part in construction of the fuzzy model are shown as follows,



We combine them in such a way that the probabilistic characteristics form an underlying structure of the fuzzy equation of the model. Take composition of fuzzy sets and relation which is performed with regard to these elements of $\mathbb{U}$ and $\mathbb{X}$ that have "enough strong" probabilistic ties. This leads to the following modification of max-min composition,

$$X_{k+1}(y_i)=\max_{u_l\in\mathbb{U},\,x_j\in\mathbb{X}\,:\,p(y_i|x_j,u_l)\geq b}\left[U_k(u_l)\wedge X_k(x_j)\wedge R(u_l,x_j,y_i)\right] \qquad /5/$$

$y_i\in\mathbb{X}$, $\wedge=\min$. "b" denotes a threshold level eliminating elements of $\mathbb{U}\times\mathbb{X}$ that are not strongly tied with the specified element of the space of state $\mathbb{X}$, $b\in[0,1]$. Higher values of "b" imply stronger influence of the probabilistic layer on the model discussed. For b=0.0 this layer is neglected. For this modification we use the following notation,

$$X_{k+1}=(U_k\circ X_k)\underset{p}{\circ} R\,, \qquad /6/$$

the used index "p" underlines the imposed probabilistic structure. Let us introduce a Boolean relation defined pointwise as

$$B(u_l,x_j,y_i;b)=\begin{cases}1, \text{if } p(y_i|x_j,u_l)\geq b\\ 0, \text{otherwise}\end{cases} \qquad /7/$$

It makes it possible to rewrite /5/ as equal to,

$$X_{k+1}(y_i)=\max_{x_j\in\mathbb{X},\,u_l\in\mathbb{U}}\left[U_k(u_l)\wedge X_k(x_j)\wedge R(u_l,x_j,y_i)\wedge B(u_l,x_j,y_i;b)\right] \qquad /8/$$

$y_i\in\mathbb{X}$.

This leads to notation,

$$X_{k+1}=(U_k\circ X_k)\underset{p}{\circ} R=(U_k\circ X_k)\bullet B\circ R \qquad /9/$$

It is worthwile to mention that the fuzzy set X_{k+1} is indexed by the values of the threshold level "b",. One has a straightforward expression,

$b_1<b_2 \Rightarrow X_{k+1}(b_1)\geq X_{k+1}(b_2)$, where $X_{k+1}(b)$ denotes the fuzzy set resulting from /8/ for the fixed value of the threshold level.

3. Calculation of the fuzzy relation of the model

At the very beginning introduce notation,

$$\mathcal{R}_k=\left\{R\in F(\mathbb{U}\times\mathbb{X}\times\mathbb{X})\;\middle|\;U_k\circ X_k\circ R=X_{k+1}\right\}. \qquad /10/$$

Sanchez provided a way of solving the fuzzy equation with the respect to the fuzzy relation R, Proposition 1 cf.[5].

If $\mathcal{R}_k \neq \emptyset$ then $\hat{R}_k = \max \mathcal{R}_k$,

$$\hat{R}_k = (U_k \circ X_k) \circledast X_{k+1} \qquad /11/$$

where

$$\hat{R}_k(u_l, x_j, y_i) = (U_k(u_l) \wedge X_k(x_j)) \alpha X_{k+1}(y_i). \qquad /12/$$

If $\hat{R} \notin \mathcal{R}_k$ then $\mathcal{R}_k = \emptyset$.

Further on let us consider the set of equations

$$U_k \circ X_k \circ R = X_{k+1}, k=1,2,\ldots,K. \qquad /13/$$

Then one has

Proposition 2

If $\bigcap_{k=1}^{K} \mathcal{R}_k \neq \emptyset$ then $\hat{R} \in \mathcal{R}_k$ for all $k=1,2,\ldots,K$,

$$\hat{R} = \bigcap_{k=1}^{K} ((U_k \circ X_k) \circledast X_{k+1}). \qquad /14/$$

The second proposition has a direct application in determination of the fuzzy relation of the fuzzy model when a collection of fuzzy data is given

$$\begin{matrix} U_1 & X_1 & X_2 \\ U_2 & X_2 & X_3 \\ & \vdots & \\ U_K & X_K & X_{K+1} \end{matrix} \qquad /15/$$

Note only that the fuzzy relation $\hat{R}$ computed according to /14/ requires satisfaction of the assumption that is hard to expect.One has to pay attention that randomness,even it happens with the fuzzy data,has not been taken into account while solving the equation.It is of interest that probabilistic sets lead to a certain weak assumption [2].

A clue point of determination of the model relation lies in fact that the probabilistic layer helps to eliminate some inconsistent fuzzy data/inconsistent with respect to the statistical knowledge of the system/.Computation of the fuzzy relation comes according to a sequence of steps,

Fix the values of "b".

1.Put $\hat{R}=$ [illegible]/all the elements of the relation are equal to 1/,

2.Put $k=1$,

3.Calculate $\circledast$-composition of $U_k \circ X_k$ and X_{k+1} and check whether

$$(U_k \circ X_k) \underset{p}{\circ} ((U_k \circ X_k) \circledast X_{k+1}) = X_{k+1} \qquad /16/$$

is satisfied,

4.If so,modify $\hat{R}$ taking intersection of the previous $\hat{R}$ and $(U_k \circ X_k) \circledast X_{k+1}$,

. Increase k, $k=k+1$, if $k \leq K$ go to 3, otherwise stop.

The value of the threshold level "b" is chosen in order to minimize a loss function Q,

$$Q=\sum_{k=1}^{K} d\left(\left(U_k \circ X_k\right) \underset{p}{\circ} R, X_{k+1}\right) \qquad /17/$$

where d stands for any distance measure 1 between the fuzzy sets.

Example. This simple numerical example explains the basic idea proposed. Consider the Boolean data set with frequencies of occurrence/n_k/,

no.	X_k		Y_k		n_k
1	1	0	0	1	1
2	1	0	1	0	N
3	0	1	1	0	1
4	0	1	0	1	N

This implies the fuzzy model of the structure,

$$Y= X \underset{p}{\circ} R \qquad /18/$$

Calculation of a matrix of conditional probabilities yields,

$$\mathbb{P} = \left[p(y_j | x)\right] = \begin{bmatrix} \frac{N}{N+1} & \frac{1}{N+1} \\ \frac{1}{N+1} & \frac{N}{N+1} \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_1 \end{bmatrix}$$

The proposed algorithm gives us the following results,

$$b<p_2 \qquad \hat{R} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad Q=2\,(N+1)$$

$$b \gg p_2 \qquad \hat{R}= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q=2\ .$$

The proper value of the threshold level higher then p_2 eliminates the subset of data set/no.1 and 3/, as inconsistent with the probabilistic layer.

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